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Ambiguity and the Home Currency Bias

Urban Ulrych
University of Zurich and Swiss Finance Institute

Nikola Vasiljevic
University of Zurich

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Urban Ulrych[a,b] Nikola Vasiljević[a]

urban.ulrych@bf.uzh.ch nikola.vasiljevic@bf.uzh.ch

[a] *Department of Banking and Finance, University of Zürich*

[b] *Swiss Finance Institute*

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Abstract

This paper addresses the question of optimal currency exposure for a risk-and-ambiguity-averse international investor. A robust mean–variance model with smooth ambiguity preferences is used to derive the optimal currency exposure. In the theoretical part, we show that the sample-efficient currency demand can be calculated as the solution to a generalized ridge regression. Through the lens of these results, we demonstrate that our ambiguity-based model offers a new explanation of the home currency bias. The investor's dislike for model uncertainty induces a disproportionately high currency hedging demand. The empirical analysis of currency overlay strategies employs the foreign exchange, equity, and bond returns over the period from 1999 to 2018. Our out-of-sample back-tests illustrate that accounting for ambiguity enhances the stability of estimated optimal currency exposures and significantly improves the portfolio performance net of transaction costs.

Keywords: Ambiguity aversion, model uncertainty, optimal currency overlay, generalized ridge regression, international asset allocation.

JEL Classification: D81, D83, F31, G11, G15.

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1 Introduction

Diversification is often said to be the only free lunch in finance. International asset allocation is a natural way to improve risk-adjusted portfolio performance—it opens access to a multitude of global investment opportunities and allows diversification across asset classes, factors, styles and geographies. A carefully tailored international portfolio can improve both growth potential and risk management. However, risks can only be reduced and not eliminated. One of the main challenges for global asset allocation is the currency risk. Fluctuations of exchange rates are influenced by macroeconomic, financial, and political factors. Shifts in currency markets can have a profound impact on portfolio performance. Therefore, investors have to decide on the hedging policy in their portfolios, i.e., the amount of foreign currency exposure that should be hedged.

The question of currency hedging has generated a lot of interest in academia and practice, and a number of competing studies are available in the literature. In this paper, we present a novel approach to calculate the optimal foreign currency exposure in an international portfolio. In a unified framework, we consider model uncertainty and investor's risk and ambiguity preferences, and solve a robust mean–variance optimization problem for currency overlay strategies. To the best of our knowledge, this is the first paper that addresses optimal currency hedging under model uncertainty and ambiguity.

Adding the ambiguity component into the decision of optimal currency exposure is valuable for at least two reasons. First, there is a general dissatisfaction with the empirical and predictive performance of the rational expectations paradigm and expected utility theory. The literature on ambiguity is currently perceived as one of the most promising avenues for future research. Our model makes a contribution in this direction as we explicitly incorporate the uncertainty associated with predictive models. Second, several papers analyzed the key events in financial markets during the Global Financial Crisis of 2008–2009 (GFC). One of the main messages is that these events are connected to ambiguity in the form of poorly understood information and are related to investors' aversion to difficult-to-quantify uncertainty, as opposed to risk. More recently, the unpegging of the Swiss franc, Brexit, US–Sino trade tensions, and other political events, fueled risk and uncertainty in the foreign exchange market.

Another important aspect of our work is that we explicitly consider currency overlay strategies in a portfolio context. Such strategies are widely applied in asset and wealth management industry, and they are designed for a specific purpose—to separate asset allocation from currency hedging decisions. Arguably, this approach improves the risk management process. Following this idea, we assume throughout this paper that portfolio weights are pre-defined, and focus solely on the optimization of foreign currency exposure. This allows us to study the impact of ambiguity on the home currency bias. Investors have a tendency to hedge significant portions of their currency exposure, very often beyond the optimal levels. Our paper proposes an ambiguity-aversion-based explanation of this behavior, and therefore contributes to otherwise scarce literature on the home currency bias.

The paper is organized as follows. Section 3 introduces a theoretical framework for international asset allocation. Our approach is model-free in the sense that no assumptions are imposed on the dynamics of asset and currency returns. However, currency models can be easily embedded in the framework. We employ a robust mean–variance model which accounts for smooth ambiguity preferences, and solve the currency overlay optimization problem for a risk-and-ambiguity-averse investor. Under certain assumptions, we show that the optimal in-sample currency exposure can be interpreted as the solution to a generalized ridge regression, and therefore has an explicit geometric interpretation. The equivalence between the robust mean–variance optimization problem and the generalized ridge regression is particularly interesting since it establishes a connection between financial economics and statistical learning. Section 4 presents an empirical study in which we investigate the effect of the ambiguity aversion on the optimal currency exposure. We demonstrate that ambiguity induces an increase in bias in parallel with a shrinkage of the confidence intervals of the proposed estimator. This result gives rise to a possible bias–variance trade-off. Thus, the efficiency of the optimal exposure estimator can be improved by considering the effect of model uncertainty directly in investor's utility function. We conclude our empirical analysis by conducting an out-of-sample back-test. The data set spans the period from 1999 until 2018. We show that dynamic hedging strategies based on our model outperform static hedging rules net of transaction costs. Moreover, accounting for ambiguity induces stability in optimal exposure estimates and reduces hedging turnover. Section 5 concludes. Additional proofs and figures are presented in the Appendix.

2 Literature Review

A number of currency hedging strategies have been presented in the literature. Starting already with the seminal papers that aimed to address this research question, opinions among different authors were divided. Perold and Schulman (1988) proposed 100% hedging as the optimal strategy. They argued that since a currency trade is a zero-sum game, the hedging reduces volatility without loss of expected return over the long run. Froot (1993), on the other hand, concluded that full hedging can reduce risk over short horizons, however, over the long term it may actually increase risk without an adequate return compensation. More specifically, in the long run, hedged returns are dominated by surprises in inflation and real interest rates. Therefore, hedging currency exposure does not provide a protection against risk factors affecting long-term exchange rates, and the hedging ratio should be zero. Under certain (strong) assumptions, Black (1989) showed that all investors should apply a universal hedging policy, irrespective of the portfolio composition and the reference currency. Glen and Jorion (1993) showed that international diversification decreases portfolio risk whether or not the assets are hedged. Although these papers fundamentally disagree about the optimal hedging policy, they do share one common trait—their results represent extreme outcomes and can be seen as corner solutions of the optimal currency exposure problem.

Other researchers obtained rather mixed results. Solnik (1993) found that, in the short term, the optimal currency hedging is specific for each investor. The proposed hedging policy is a function of the portfolio structure and the percentage of foreign assets. More recent literature shares a broader consensus that currency hedging tends to lower the portfolio volatility. Additionally, conditional hedging outperforms strategies that employ fixed hedge ratios. Haefliger, Wydler, and Waelchli (2002) proposed full hedging for fixed-income portfolios, while equity portfolios can only be partially hedged (or even unhedged), depending on the correlations between equity and currency returns. Jorion (1994) considered a global mean–variance optimization, where positions in assets and currencies can be determined simultaneously or separately. Either way, the optimal currency exposure depends on the portfolio reference currency. As shown by Eun and Resnick (1988), a low accuracy of estimated input parameters—in particular the mean returns—is the main driver of poor ex-ante performance of the joint optimization of asset allocations and currency hedge ratios. On the other hand, hedging for the purpose of risk minimization mitigates estimation risk, as the covariance structure is found to be estimated with higher precision. Schmittmann (2010) analyzed constant hedging strategies in comparison to the static variance minimizing hedging ratios calculated with ordinary least squares, while De Roon, Eiling, Gerard, and Hillion (2011) included currency positions as a further asset class and pointed out that risk hedging and speculative benefits are two motivations for internationally diversified portfolios. Overall, the studies provide supporting evidence showing that currency hedging reduces risk in multi-currency portfolios.

The seminal work in this area is arguably the paper *Global Currency Hedging* of Campbell, Serfaty-De Medeiros, and Viceira (2010). The authors proved that it is possible to find optimal hedging ratios minimizing volatility for general portfolios. Such hedging decision should be made considering the correlations between currencies and equities. The authors use data over the period of 1975 to 2005 and empirically show that the US dollar (particularly in relation to the Canadian dollar), the euro, and Swiss franc (particularly in the second half of the analyzed period) have moved against world equity markets. Thus, these currencies should be attractive to risk-minimizing global equity investors despite their low average returns. A long position in the US-Canadian exchange rate is a particularly effective hedge against equity risk. These results hold for both short and long-term investment horizons. On the other hand, the authors show that most currency returns are almost uncorrelated with bond returns. Thus, risk-minimizing bond investors should avoid holding currencies; that is, they should fully currency-hedge their international bond positions. This is consistent with common practice of institutional investors.

It is important to notice that all of the above noted works in the currency hedging literature assume that investors know perfectly the true probability law governing the stochastic processes of asset and currency returns. However, in many situations, agents are uncertain about the validity of the model, and hence any particular probability law used to describe return processes is subject to potential parameter estimation errors and model misspecification. This drawback can be addressed with the incorporation of different methods that account for decision making under ambiguity in the study of portfolio analysis. Let us review a few key definitions relating to ambiguity, its difference from risk, and how aversion to ambiguity might be measured. The two most prominent approaches of how aversion to uncertainty might be measured and modelled are the use of Bayesian portfolio analysis and the use of ambiguity averse preferences. Bayesian portfolio analysis employs and facilitates the use of fast, intuitive, and easily implementable numerical algorithms which are able to simulate otherwise complex economic quantities. On the other hand, ambiguity averse preferences take a leap from the traditional use of rational expectations and maximization of (subjective) expected utility. Uncertainty about the predictions is derived and captured directly in the utility function and in this way decision making is performed under ambiguity about the financial market outcomes, such as portfolio choices and equilibrium asset prices.

That a distinction might be drawn between standard expected utility and more general decision models has been known since Knight (1921). According to Knight, there are two kinds of uncertainty: the first, called risk, corresponds to situations in which all relevant events are associated with a (objectively or subjectively) uniquely determined probability assignment; the second, called (Knightian) uncertainty, corresponds to situations in which some events do not have an obvious probability assignment. The experimental relevance of the distinction between risk and uncertainty has been formally discussed by Ellsberg (1961), whose findings have shown that agents are not always able to derive a unique probability distribution over the reference state space. After Ellsberg's seminal paper, uncertain environments have become better known as ambiguous and the general dislike for them as ambiguity aversion.

Let us start with the review of the literature of Bayesian portfolio studies. Although recognized by Markowitz (1952), the problem of estimation errors did not receive serious attention until the 1970s, when the first applications in finance are entirely based on uninformative or data-based priors. Later, Jorion (1986) introduced the hyperparameter prior approach in the spirit of the Bayes-Stein shrinkage prior, whereas Black and Litterman (1992) advocated an informal Bayesian analysis with economic views and equilibrium relations. The study by Pastor (2000) centers prior beliefs around values implied by asset pricing theories, and Tu and Zhou (2010) argue that the investment objective provides a useful prior for portfolio selection. All of these studies assume that asset returns are identically and independently distributed through time, whereas Kandel and Stambaugh (1996), Barberis (2000), and Avramov (2002) account for the possibility that returns are predictable by macro variables such as the aggregate dividend yield, the default spread, and the term spread. Moreover, Uppal and Wang (2003) note that the evidence from experimental economics and psychology suggest that, in some situations, agents' uncertainty cannot be expressed using a single probability distribution; that is, ambiguity is present. Uppal and Wang develop a model of intertemporal portfolio choice in which investors account explicitly for this ambiguity. Building on this, Garlappi, Uppal, and Wang (2007) present a model for an investor with multiple priors and aversion to ambiguity. Multiple priors are characterized by a confidence interval around the estimated parameters and the ambiguity aversion is modelled via a minimization over the priors.

The other strand of literature captures the ambiguity aversion directly in the decision maker's preferences and consequently also in the utility function itself. When a decision maker has too little information to form a single prior, she may plausibly consider a set of probability distributions and not a unique prior. Schmeidler (1989) formalized this intuition starting from the observation that the probability attached to an uncertain event may not reflect the heuristic amount of information that has led to that particular probability assignment. Motivated by this consideration, Schmeidler suggested to assign non-additive probabilities, or capacities, to allow for the encoding of information that additive probabilities cannot represent. Gilboa and Schmeidler (1989) extended this work by presenting multiple prior preferences. After the seminal paper by Gilboa and Schmeidler had gained popularity, Anderson, Hansen, and Sargent (1998, 2003) and Hansen and Sargent (2001) noted that multiple-prior criteria also appear in the robust control theory used in engineering. Robust control theory specifies the set of probabilities by taking a single "approximating model" and statistically perturbing it. This reflects a situation wherein agents have a specific model of reference and, acknowledging the possibility of errors, seek robustness against misspecifications.

Klibanoff, Marinacci, and Mukerji (2005) proposed that the ambiguity of a risky event can be characterized by a set of subjectively plausible cumulative probability distributions for this event. The decision maker subjectively weights these distributions and resulting (KMM) preference relation describes the investor's attitude towards ambiguity. Building on this, Maccheroni, Marinacci, and Ruffino (2013) derive the analogue of the classic Arrow-Pratt approximation of the certainty equivalent (given the underlying KMM preferences) under model uncertainty as described by the smooth model of decision making under ambiguity. They study its scope by deriving a tractable mean–variance model adjusted for ambiguity and solving the corresponding portfolio allocation problem. We use a similar approach in this work and connect this ambiguity aversion adjusted mean–variance preferences to the optimal currency exposure framework. The analytical tractability of the enhanced Arrow-Pratt approximation renders this model especially well suited for calibration exercises aimed at exploring the consequences of model uncertainty on the optimal currency allocations.

Let us have a final note of caution on terminology. In the literature, ambiguity and uncertainty are not always clearly distinguished, nor are they clearly defined. In this paper, we use both terms equivalently. Uncertainty or ambiguity is meant to represent "non-probabilized" uncertainty (situations in which the decision maker is not given probabilistic information about the external events that affect the outcome of a decision) as opposed to risk, which is "probabilized" uncertainty.

3 Theoretical Framework

In this section, we first consider currency hedging in an international portfolio context. In particular, we derive results for model-free hedged and unhedged portfolio returns using currency forwards. Our approach allows for inclusion of cross-hedging strategies with currencies to which investor's portfolio is not directly exposed. In addition to the general results, we demonstrate how a currency model could be embedded in our framework. In a second step, we summarize Maccheroni, Marinacci, and Ruffino (2013)'s robust mean–variance optimization, and derive optimal currency overlay strategies for a risk-and-ambiguity-averse investor. Further, we show that a generalized ridge regression approach can be used to recover sample-efficient currency exposures.

3.1 Currency Hedged and Unhedged Portfolio Returns

Let us consider an asset i whose price in local currency at time t is denoted by $P_{i,t}$. The simple return on this asset from t to $t+1$ is given by $R_{i,t+1}$.[1] We introduce $S_{c,t}$ as the spot exchange rate in the reference/home currency (HC) per unit of foreign/local currency (LC) at time t, where c denotes the local currency of asset i. Alternatively, for financial assets which have embedded multi-currency exposure (e.g., global equity indices) c_i can be interpreted as a single currency in which prices of asset i are quoted on markets. The exchange rate return from t to $t+1$ is $e_{c,t+1}$. Further, let us assume that an investor holds $\lambda_{i,t}$ shares of stock i at time t, or an equivalent number of fixed-income instruments, commodity contracts or other investment positions.[2] The unhedged return on asset i expressed in the home currency is

$$\tilde{R}^{u}_{i,t+1} = \frac{\lambda_{i,t+1} P_{i,t+1} S_{c,t+1}}{\lambda_{i,t} P_{i,t} S_{c,t}} - 1 = \underbrace{R_{i,t+1}}_{LC\ asset\ return} + \underbrace{e_{c,t+1}}_{FX\ return} + \underbrace{R_{i,t+1} e_{c,t+1}}_{cross-product}. \tag{1}$$

This result demonstrates that the unhedged asset return in home currency is driven by three components: (a) The asset return in local currency, (b) The exchange rate return, and (c) The second-order

[1] The following notational convention is applied throughout the paper. Local currency returns are denoted by $R_{\star,\star}$, where the first (second) subscript indicates the investment position (time). Home currency returns are denoted by $\tilde{R}^{\star}_{\star,\star}$. The subscripts bear the same meaning as for the local currency returns, whereas the superscript indicates currency hedging. To ease notation we drop indexation of the home currency in mathematical expressions. All results derived in this paper hinge on the assumption that a home currency is prespecified.

[2] To ensure that returns are influenced only by market fluctuations and not by rebalancing, we have assumed that $\lambda_{i,t} = \lambda_{i,t+1}$.

cross-product between the first two terms.

3.1.1 Unhedged Portfolio Returns

We now move from a single-asset case to a portfolio context. Consider an investor with an arbitrary home currency and invested in a portfolio $\mathcal{P}$ which consists of N assets. The fraction of wealth invested in asset $i = 1, 2, \ldots, N$ is defined as $x_{i,t}$. Let us further assume that this portfolio has a direct exposure to K foreign currencies. For simplicity, we label the home currency by $c = 1$, whereas the foreign currencies are denoted by $c = 2, 3, \ldots, K + 1$. The assets can be classified and grouped by their local currency. The collection of all assets denominated in currency c held in a portfolio at time t is denominated by $\mathcal{A}_{c,t}$. The fraction of wealth directly exposed to currency c is defined as $w_{c,t} := \sum_{j \in \mathcal{A}_{c,t}} x_{j,t} \neq 0$. We introduce $R_{\mathcal{P}_c,t+1} := \sum_{j \in \mathcal{A}_{c,t}} \frac{x_{j,t}}{w_{c,t}} R_{j,t+1}$ as the return on sub-portfolio $\mathcal{P}_c$ which consists of all assets denominated in currency c. From Eq. (1) it follows that the unhedged return on sub-portfolio $\mathcal{P}_c$ expressed in an investor's home currency can be computed as

$$\tilde{R}^u_{\mathcal{P}_c,t+1} = R_{\mathcal{P}_c,t+1} + e_{c,t+1} + R_{\mathcal{P}_c,t+1} e_{c,t+1}. \tag{2}$$

Eq. (1)–(2) usher in two representations for the unhedged return on portfolio $\mathcal{P}$:

$$\tilde{R}^u_{\mathcal{P},t+1} = \sum_{i=1}^{N} x_{i,t} \tilde{R}^u_{i,t+1} = \sum_{c=1}^{K+1} w_{c,t} \tilde{R}^u_{\mathcal{P}_c,t+1}, \tag{3}$$

with $\sum_{i=1}^{N} x_{i,t} = \sum_{c=1}^{K+1} w_{c,t} = 1$, for every t. By substituting Eq. (2) in the second representation in Eq. (3) we obtain the following result:

$$\tilde{R}^u_{\mathcal{P},t+1} = \underbrace{\sum_{c=1}^{K+1} w_{c,t} R_{\mathcal{P}_c,t+1}}_{LC\ sub-portfolio\ returns} + \underbrace{\sum_{c=2}^{K+1} w_{c,t} e_{c,t+1}}_{FX\ returns} + \underbrace{\sum_{c=2}^{K+1} w_{c,t} R_{\mathcal{P}_c,t+1} e_{c,t+1}}_{cross-products}. \tag{4}$$

3.1.2 Hedged Portfolio Returns

We consider currency overlay strategies which are implemented via forward exchange contracts. The forward exchange rate in home currency per unit of foreign currency c at time t is denoted by $F_{c,t}$.[3] We consider the forward contract with delivery date $t + 1$. The forward premium, i.e., the return on the forward contract, is defined as $f_{c,t} := (F_{c,t} - S_{c,t})/S_{c,t}$. At time t, this quantity is known and represents the cost of carry.[4]

We denote by $\phi_{c,t}$ the amount invested at time t in a forward exchange contract for currency c, expressed in the home currency as a fraction of total portfolio value. Similarly, $\phi_{c,t}/S_{c,t}$ represents the

[3] The price of the forward contract is assumed to be zero at the inception.

[4] We note that $F_{1,t} = 1$ and $f_{1,t} = 0$ trivially hold for all t.

relative notional value of the forward contract in local currency c at time t. Hedging of foreign currency exposure can be achieved by selling a forward exchange contract (i.e., $\phi_{c,t} > 0$). By no arbitrage principle this is analogous to shorting foreign bonds and holding domestic bonds.[5] The pay-off at time $t+1$ is $F_{c,t} - S_{c,t+1}$.

We extend the investment universe by assuming that the total number of foreign currencies which are available on the market is $M \geq K$. Therefore, an investor could trade in currencies to which her portfolio is not directly exposed. This creates an opportunity to implement cross-hedging strategies which are common in wealth and asset management industry. A hedged portfolio return is then equal to

$$\tilde{R}^h_{\mathcal{P},t+1} = \tilde{R}^u_{\mathcal{P},t+1} + \sum_{c=2}^{M+1} \phi_{c,t}(f_{c,t} - e_{c,t+1}), \tag{5}$$

where $f_{c,t} - e_{c,t+1} = (F_{c,t} - S_{c,t+1})/S_{c,t}$ represents the normalized payoff of a short forward contract on currency c at time $t+1$. Since $S_{1,t} = F_{1,t} = 1$ for all t, the choice of $\phi_{1,t}$ is completely arbitrary. Nevertheless, we would like to keep the interpretation of $\phi_{c,t}$ as a fraction of total portfolio value corresponding to the notional of the forward contract on currency c. Therefore, we impose the following condition:

$$\phi_{1,t} = 1 - \sum_{c=2}^{M+1} \phi_{c,t}. \tag{6}$$

Consequently, all (net) currency exposures sum up to zero, i.e., the currency portfolio is a zero investment portfolio. For a portfolio which is directly exposed to currency c (i.e., meaning $w_{c,t} \neq 0$), the hedge ratio can be defined as $h_{c,t} := \phi_{c,t}/w_{c,t}$. If $\phi_{c,t} = h_{c,t} = 0$, the assets denominated in currency c are unhedged. Conversely, if $\phi_{c,t} = w_{c,t}$ or $h_{c,t} = 1$, the assets are fully hedged.

Fully hedged returns can be computed by setting $\phi_{c,t} = w_{c,t}$ for $c = 1, 2, \ldots, K+1$, and $\phi_{c,t} = 0$ for $c = K+2, K+3, \ldots, M+1$ in Eq. (5). Cross-hedging with additional currencies is excluded in this case. Therefore, the fully hedged portfolio return can be expressed as

$$\tilde{R}^{fh}_{\mathcal{P},t+1} = \underbrace{\sum_{c=1}^{K+1} w_{c,t} R_{\mathcal{P}_c,t+1}}_{LC\ returns} + \underbrace{\sum_{c=2}^{K+1} w_{c,t} f_{c,t}}_{FX\ forward\ premia} + \underbrace{\sum_{c=2}^{K+1} w_{c,t} R_{\mathcal{P}_c,t+1} e_{c,t+1}}_{cross-products}. \tag{7}$$

Comparison of this result with the expression for unhedged portfolio returns in Eq. (4) reveals that currency returns are replaced by forward premia in the case of a fully hedged portfolio. Therefore, currency hedging eliminates randomness stemming from the exchange rate fluctuations. This is due to the fact that—for any foreign currency c—the forward premium $f_{c,t}$ is known at time t. However, the exchange rate risk is not completely eliminated since the second-order cross-product term remains. The hedging positions are entered at time t, and the exact currency exposure at time $t+1$ is not known

[5] Alternatively, this is equivalent to borrowing funds in foreign currency and lending the same amount in home currency.

in advance.[6]

Unhedged and fully hedged portfolio returns represent two special cases of interest, which lie on the opposite sides of the currency hedging spectrum. To accommodate partial hedging of direct currency exposure and cross-hedging with additional currencies, we define the net exposure to currency c as

$$\psi_{c,t} \coloneqq w_{c,t} - \phi_{c,t}. \tag{8}$$

Here, $w_{c,t}$ represents the direct currency exposure and $\phi_{c,t}$ reflects the position in a forward contract. We distinguish among several cases of currency hedging:

(a) No hedging: $\psi_{c,t} = w_{c,t}$,

(b) Full hedging: $\psi_{c,t} = 0$,

(c) Partial hedging: $0 < \psi_{c,t} \leq w_{c,t}$,

(d) Over-hedging and under-hedging: $\psi_{c,t} < 0$ and $\psi_{c,t} > w_{c,t}$, respectively.

Cross-hedging strategies with the remaining $M - K$ currencies can be implemented through forward contracts. For $c = K+2, K+3, \ldots, M+1$, the direct currency exposure is zero (i.e., $w_{c,t} = 0$), and the net exposure is $\psi_{c,t} = -\phi_{c,t}$.

Eq. (5) can be rewritten in the following form:

$$\begin{aligned}\tilde{R}^{h}_{\mathcal{P},t+1} &= \tilde{R}^{fh}_{\mathcal{P},t+1} + \sum_{c=2}^{K+1} \psi_{c,t}(e_{c,t+1} - f_{c,t}) - \sum_{c=K+2}^{M+1} \phi_{c,t}(e_{c,t+1} - f_{c,t}) \\ &= \tilde{R}^{fh}_{\mathcal{P},t+1} + \sum_{c=2}^{M+1} \psi_{c,t}(e_{c,t+1} - f_{c,t}).\end{aligned} \tag{9}$$

Additionally, from Eq. (6) we compute the net home currency exposure:

$$\psi_{1,t} = -\sum_{c=2}^{M+1} \psi_{c,t}. \tag{10}$$

Eq. (5) and (9) are mathematically equivalent, however they offer different economic interpretations. The former (latter) equation decomposes portfolio returns into unhedged (fully hedged) returns and a currency hedging (net currency exposure) component.

[6] Over tactical investment horizons and in most market environments, the cross-product term is relatively small and can be neglected. Hence, nearly perfect currency hedging is typically achieved in practice. However, over longer investment horizons (e.g., beyond one year) or in the case of a sudden market crash, these terms could have a significant impact on portfolio returns. To address this issue, one possible solution is to review currency hedging policies when market conditions change substantially.

3.1.3 Embedding Currency Models: An Example

All results derived above are model-free. They represent an essential cornerstone on top of which a modelling framework can be superposed. To illustrate this, we consider a particularly relevant case in point—the covered interest rate parity (CIRP) model. Let us denote the nominal risk-free interest rate in currency c by $r_{c,t}$. Based on the notation introduced above, $r_{1,t}$ represents the risk-free interest rate in the home currency, whereas other risk-free interest rates correspond to the foreign currencies. The covered interest parity asserts that $F_{c,t}/S_{c,t} = (1 + r_{1,t})/(1 + r_{c,t})$. Consequently, the forward premium can be computed as $f_{c,t} = (r_{1,t} - r_{c,t})/(1 + r_{c,t}) \approx r_{1,t} - r_{c,t}$.[7] Plugging this approximation into Eq. (9) yields the following result for hedged excess portfolio returns:

$$\tilde{R}^h_{\mathcal{P},t+1} - r_{1,t} \approx \underbrace{\sum_{c=1}^{K+1} w_{c,t}(R_{\mathcal{P}_c,t+1} - r_{c,t})}_{LC\ excess\ sub-portfolio\ returns} + \underbrace{\sum_{c=2}^{M+1} \psi_{c,t}(e_{c,t+1} - r_{1,t} + r_{c,t})}_{excess\ exchange\ rate\ return} + \underbrace{\sum_{c=2}^{K+1} w_{c,t} R_{\mathcal{P}_c,t+1} e_{c,t+1}}_{cross-products}.$$

Therefore, the total return can be decomposed into three components: (a) Allocation-weighted average of the excess local-currency returns on sub-portfolios $\mathcal{P}_c$, (b) Net-exposure-weighted average of CIRP payoffs, and (c) Allocation-weighted average of cross-products between the local-currency returns on sub-portfolios $\mathcal{P}_c$ and the corresponding exchange rate returns.

3.2 Smooth Ambiguity Preferences and Robust Mean–Variance Optimization

This section presents a robust mean–variance setting for decision making which accounts for model uncertainty. First, we outline the mathematical framework for our optimization problem.[8] We then summarize Maccheroni, Marinacci, and Ruffino (2013)'s results for the Arrow-Pratt approximation of the certainty equivalent in the case of a risk-and-ambiguity-averse agent who maximizes von Neumann-Morgenstern expected utility. This model is the workhorse of our analysis.

We define a state space Ω, which consists of all possible realizations of uncertainty. Sets of states of nature are called events ω, and the outcome space represents a σ-algebra $\mathcal{F}$ which contains random payoffs of agent's decisions. A preference relation is defined over the mapping from Ω to $\mathcal{F}$. In a risk-only setting, all agents agree on the probability measure $\mathbb{P}$. Ambiguity introduces the space Δ of possible models $\mathcal{Q}$ in order to capture model uncertainty effects. Let us assume that an agent's prior over all probability measures $\mathbb{Q}$ corresponding to the models in $\mathcal{Q}$ is given by μ. Then, we can compute the reduced probability $\bar{\mathbb{Q}} := \int_\Delta \mathbb{Q}\, d\mu(\mathbb{Q})$ induced by the prior μ—also called the barycenter of μ—which plays an important role in our setting.[9]

[7] We neglect the second-order cross-product term since it has a minuscule effect on the results.
[8] We provide only a heuristic overview. For technical details see Maccheroni, Marinacci, and Ruffino (2013).
[9] The probability measure $\bar{\mathbb{Q}}$ is called a reduction of μ on Ω since it can be interpreted in terms of reduction of compound lotteries. For example, if $\text{supp}(\mu) = \{\mathbb{Q}_1, \mathbb{Q}_2, \ldots, \mathbb{Q}_n\}$ is finite and $\mu(\mathbb{Q}_i) = \mu_i$ for $i = 1, 2, \ldots, n$, then $\bar{\mathbb{Q}}(A) = \mu_1\mathbb{Q}_1(A) + \mu_2\mathbb{Q}_2(A) + \cdots + \mu_n\mathbb{Q}_n(A)$, for any event (set of outcomes) $A \in \mathcal{F}$.

To account for model uncertainty, Maccheroni, Marinacci, and Ruffino (2013) proposed a robust optimization which builds on the classical mean–variance expected utility framework and embeds smooth ambiguity model of Klibanoff, Marinacci, and Mukerji (2005). The utility function is given by

$$U(\ell) = \mathrm{E}_{\bar{\mathbb{Q}}}[\,\ell\,] - \frac{\lambda}{2}\mathrm{Var}_{\bar{\mathbb{Q}}}[\,\ell\,] - \frac{\theta}{2}\mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\ell\,]\,], \tag{11}$$

where ℓ is an uncertain prospect, and positive coefficients λ and θ represent risk and ambiguity aversion, respectively. Terms $\mathrm{E}_{\bar{\mathbb{Q}}}[\,\cdot\,]$ and $\mathrm{Var}_{\bar{\mathbb{Q}}}[\,\cdot\,]$ are the reduced-probability estimators of mean and variance obtained by combining predictions from different $\mathcal{Q}$-models. The underlying weighting scheme is captured by the agent's prior μ. The estimator $\mathrm{Var}_{\bar{\mathbb{Q}}}[\,\cdot\,]$ measures risk, whereas the estimator $\mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\cdot\,]\,]$ quantifies the model uncertainty (i.e., the dispersion of different $\mathcal{Q}$-predictions with respect to μ). It can be computed as

$$\mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\ell\,]\,] = \int_{\Delta}\left(\int_{\Omega}\ell(\omega)\,d\mathbb{Q}(\omega)\right)^{2} d\mu(\mathbb{Q}) - \left(\int_{\Delta}\left(\int_{\Omega}\ell(\omega)\,d\mathbb{Q}(\omega)\right)d\mu(\mathbb{Q})\right)^{2}. \tag{12}$$

Higher estimates of model uncertainty indicate investor's lower confidence in a single model. Conversely, if the investor's prior μ is a singleton, the prospect is regarded as purely risky. All probability measures induced by the set of models $\mathcal{Q}$ are then mapped to the probability measure $\mathbb{P}$, which corresponds to the risk-only setting. Furthermore, the third term in Eq. (11) disappears in that case, and the model collapses into the classical mean–variance utility.

Traditional Bayesian approaches deal with estimation errors, however the main drawback is that they typically assume ambiguity-neutral investors. This information is fully captured in the reduced probability $\bar{\mathbb{Q}}$. Maccheroni, Marinacci, and Ruffino (2013) provide additional modelling flexibility in a computationally tractable and economically meaningful way. Their robust mean–variance preferences distinguish between investor's risk aversion and ambiguity aversion. This is explicitly captured in Eq. (11) with two distinct variance penalties, which correspond to the risk and model uncertainty, respectively.

3.3 Optimal Currency Overlay with Model Uncertainty and Ambiguity

Let us consider a risk-and-ambiguity-averse investor who wants to optimize currency exposure in her portfolio. The asset weights are assumed to be predefined and are not affected by the investor's decision regarding the optimal currency overlay. Such strategies are very common in practice. They are specifically designed for the management of the currency risk in portfolios—the optimal currency exposure is treated as a separate decision from the asset allocation problem. Therefore, the investor's objective is to maximize her utility function, i.e.,

$$\max_{\boldsymbol{\Psi}_t} U(\tilde{R}^h_{\mathcal{P},t+1}) = \max_{\boldsymbol{\Psi}_t} \left\{ \mathrm{E}_{\bar{\mathbb{Q}}}[\, \tilde{R}^h_{\mathcal{P},t+1} \,] - \frac{\lambda}{2} \mathrm{Var}_{\bar{\mathbb{Q}}}[\, \tilde{R}^h_{\mathcal{P},t+1} \,] - \frac{\theta}{2} \mathrm{Var}_{\mu}[\, \mathrm{E}_{\mathbb{Q}}[\, \tilde{R}^h_{\mathcal{P},t+1} \,]\,] \right\}. \tag{13}$$

Hedged portfolio returns $\tilde{R}^h_{\mathcal{P},t+1}$ are computed in Eq. (9) and represent a risky and ambiguous prospect in our robust mean–variance optimization problem. The utility function is maximized with respect to the M-dimensional vector of net foreign currency exposures $\boldsymbol{\Psi}_t := (\psi_{2,t}, \psi_{3,t}, \ldots, \psi_{M+1,t})'$. This means that our model includes both hedging of direct currency exposure in the portfolio and cross-hedging with additional currencies.

To ease notation, we introduce additional conventions. First, the vector of exchange rate returns is given by $\mathbf{e}_{t+1} = (e_{2,t+1}, e_{3,t+1}, \ldots, e_{M+1,t+1})'$. Second, the vector of forward exchange premia is $\mathbf{f}_t = (f_{2,t}, f_{3,t}, \ldots, f_{M+1,t})'$. Using the linearity of expectations and the variance sum law, we obtain the following results for the expected return, risk and model uncertainty for the investor's portfolio:

$$\mathrm{E}_{\bar{\mathbb{Q}}}[\, \tilde{R}^h_{\mathcal{P},t+1} \,] = \mathrm{E}_{\bar{\mathbb{Q}}}[\, \tilde{R}^{fh}_{\mathcal{P},t+1} \,] + \boldsymbol{\Psi}'_t \, \mathrm{E}_{\bar{\mathbb{Q}}}[\, \mathbf{e}_{t+1} - \mathbf{f}_t \,], \tag{14a}$$

$$\mathrm{Var}_{\bar{\mathbb{Q}}}[\, \tilde{R}^h_{\mathcal{P},t+1} \,] = \mathrm{Var}_{\bar{\mathbb{Q}}}[\, \tilde{R}^{fh}_{\mathcal{P},t+1} \,] + \boldsymbol{\Psi}'_t \, \mathrm{Var}_{\bar{\mathbb{Q}}}[\, \mathbf{e}_{t+1} - \mathbf{f}_t \,] \boldsymbol{\Psi}_t + 2\boldsymbol{\Psi}'_t \, \mathrm{Cov}_{\bar{\mathbb{Q}}}[\, \tilde{R}^{fh}_{\mathcal{P},t+1}, \mathbf{e}_{t+1} - \mathbf{f}_t \,], \tag{14b}$$

$$\begin{aligned} \mathrm{Var}_{\mu}[\, \mathrm{E}_{\mathbb{Q}}[\, \tilde{R}^h_{\mathcal{P},t+1} \,]\,] = {} & \mathrm{Var}_{\mu}[\, \mathrm{E}_{\mathbb{Q}}[\, \tilde{R}^{fh}_{\mathcal{P},t+1} \,]\,] + \boldsymbol{\Psi}'_t \, \mathrm{Var}_{\mu}[\, \mathrm{E}_{\mathbb{Q}}[\, \mathbf{e}_{t+1} - \mathbf{f}_t \,]\,] \boldsymbol{\Psi}_t \\ & + 2\boldsymbol{\Psi}'_t \, \mathrm{Cov}_{\mu}[\, \mathrm{E}_{\mathbb{Q}}[\, \tilde{R}^{fh}_{\mathcal{P},t+1} \,], \mathrm{E}_{\mathbb{Q}}[\, \mathbf{e}_{t+1} - \mathbf{f}_t \,]\,]. \end{aligned} \tag{14c}$$

We note that these equations prominently feature the normalized forward payoff, i.e., the difference between $\mathbf{f}_t$ and $\mathbf{e}_{t+1}$.[10] This highlights the importance of hedging decision trade-off between entering a forward contract and retaining currency exposure.

By plugging the results from Eq. (14a)–(14c) into Eq. (13), rearranging the terms and dropping those which are not a function of $\boldsymbol{\Psi}_t$, the optimization problem can be cast in a quadratic program format. Therefore, our currency overlay optimization with model uncertainty and ambiguity—including an arbitrary set of linear constraints on the currency exposures—reads as follows:

$$\begin{aligned} \underset{\boldsymbol{\Psi}_t}{\arg\min} \quad & \frac{1}{2} \boldsymbol{\Psi}'_t \mathbf{A} \boldsymbol{\Psi}_\mathbf{t} + \mathbf{b}' \boldsymbol{\Psi}_\mathbf{t} \\ \text{subject to} \quad & \mathbf{C} \boldsymbol{\Psi}_\mathbf{t} \preceq \mathbf{d}, \end{aligned} \tag{15}$$

where $\mathbf{A}$ is a $(M \times M)$-dimensional symmetric matrix and $\mathbf{b}$ is a $(M \times 1)$-dimensional vector given by

$$\mathbf{A} = \lambda \mathrm{Var}_{\bar{\mathbb{Q}}}[\, \mathbf{e}_{t+1} - \mathbf{f}_t \,] + \theta \mathrm{Var}_{\mu}[\, \mathrm{E}_{\mathbb{Q}}[\, \mathbf{e}_{t+1} - \mathbf{f}_t \,]\,], \tag{16a}$$

$$\mathbf{b} = \lambda \mathrm{Cov}_{\bar{\mathbb{Q}}}[\, \tilde{R}^{fh}_{\mathcal{P},t+1}, \mathbf{e}_{t+1} - \mathbf{f}_t \,] + \theta \mathrm{Cov}_{\mu}[\, \mathrm{E}_{\mathbb{Q}}[\, \tilde{R}^{fh}_{\mathcal{P},t+1} \,], \mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t]\,] - \mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1} - \mathbf{f}_t]. \tag{16b}$$

[10] Under the CIRP, the forward premium $f_{c,t}$ is the difference between the risk-free interest rates in foreign and home currency. Therefore, the random variable $e_{c,t+1} - f_{c,t}$ can be interpreted as the currency excess return.

An arbitrary set of L linear constrains on $\boldsymbol{\Psi}_t$ can be specified in a $(L \times M)$ matrix $\mathbf{C}$ and a $(L \times 1)$ vector $\mathbf{d}$. The operator $\preceq$ represents component-wise less-than-or-equal operator. Finally, the optimal home currency exposure can be computed using Eq. (10).

It is important to mention that linear constraints are highly relevant for practical applications. As a matter of fact, they are often based on regulatory requirements. For example, pension funds use such constraints in order to prevent extreme currency positions and therefore avoid unnecessary risks. However, adding constraints comes at a cost—an analytical solution of the constrained optimization problem is not attainable. The Kuhn–Tucker specification of the (general) optimization problem (15) yields an endogenous formulation and cannot be solved analytically. Therefore, the solution has to be obtained numerically through a quadratic programming algorithm.

3.3.1 Unconstrained Currency Overlay Optimization

The quadratic program (15) can be solved analytically when no linear constraints are imposed. Taking a derivative with respect to $\boldsymbol{\Psi}_t$ yields the first-order condition

$$-\mathbf{A}\boldsymbol{\Psi}_t - \mathbf{b} = \mathbf{0}, \tag{17}$$

where $\mathbf{0}$ is a $(M \times 1)$ vector of zeros. The Hessian matrix of second-order derivatives is $\mathrm{Hess}(\boldsymbol{\Psi}_t) = -A$. Since components of the random vector $(\mathbf{e}_{t+1} - \mathbf{f}_t)$ are linearly independent, the corresponding variance–covariance matrix $\mathrm{Var}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1} - \mathbf{f}_t\,]$ is positive definite. Furthermore, risk and ambiguity aversion coefficients (i.e., λ and θ) are positive scalars, and $\mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\mathbf{e}_{t+1} - \mathbf{f}_t\,]\,]$ is a positive semi-definite matrix. This implies that the Hessian matrix is negative definite.[11] We conclude that the first-order condition (17) is a necessary and sufficient condition to characterize the maximum of the currency overlay optimization problem (13) with respect to the vector of (unconstrained) net foreign currency exposures $\boldsymbol{\Psi}_t$. Therefore, the solution of the unconstrained optimization problem is

$$\begin{aligned}\boldsymbol{\Psi}_t^* = -&\Big(\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1} - \mathbf{f}_t\,] + \theta \mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t]\,]\Big)^{-1} \\ &\cdot\Big(\lambda \mathrm{Cov}_{\bar{\mathbb{Q}}}[\,\tilde{R}^{fh}_{\mathcal{P},t+1}, \mathbf{e}_{t+1} - \mathbf{f}_t\,] + \theta \mathrm{Cov}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\tilde{R}^{fh}_{\mathcal{P},t+1}\,], \mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t]\,] - \mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1} - \mathbf{f}_t]\Big),\end{aligned} \tag{18}$$

where the matrix in the first bracket is positive definite and its inverse exists. Due to the linear relationship between portfolio returns and currency exposures—which is a consequence of hedging with forward contracts—the objective function amounts of a sum of linear and quadratic forms in $\boldsymbol{\Psi}_t$ and the optimal solution is obtained in a closed form.

Special Case: Infinitely Risk-Averse Investor

We consider a decision maker who minimizes the variance (volatility) of returns. Such an investor is

[11] Equivalently, matrix $\mathbf{A}$ defined in Eq. (16a) is positive definite and therefore invertible.

infinitely risk-averse (i.e., $\lambda \to \infty$), and her optimal currency exposure is

$$\mathbf{\Psi}^*_{t,risk} = -\mathrm{Var}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1} - \mathbf{f}_t\,]^{-1} \cdot \mathrm{Cov}_{\bar{\mathbb{Q}}}[\,\tilde{R}^{fh}_{\mathcal{P},t+1}, \mathbf{e}_{t+1} - \mathbf{f}_t\,]. \tag{19}$$

First, we observe that in the absence of model uncertainty, the reduced probability $\bar{\mathbb{Q}}$ collapses to the probability $\mathbb{P}$ and we obtain the classical mean–variance result. Second, and irrespectively of the model uncertainty, a negative (positive) correlation between hedged portfolio returns and excess currency returns implies that the foreign currency tends to appreciate (depreciate) when the hedged portfolio loses value. Long (short) positions in such currencies are favorable, which is confirmed by positive (negative) net foreign currency exposures in (19). Moreover, if the correlation is sufficiently negative (positive), the investor could reduce portfolio risk by under-hedging (over-hedging). This can be accomplished by holding a long position in excess of the portfolio weight $w_{c,t}$ (a short position). Finally, zero correlation between hedged portfolio returns and excess currency returns implies no exposure at all. In this case currency exposure only adds risk to the investor's portfolio. An infinitely risk-averse investor cannot be appropriately compensated for the additional risk, hence she is better off if the currency exposure is fully hedged.

Special Case: Infinitely Ambiguity-Averse Investor

Let us consider now an infinitely ambiguity-averse investor, which corresponds to the case when $\theta \to \infty$. The optimal exposure is then given by

$$\mathbf{\Psi}^*_{t,amb} = -\mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\mathbf{e}_{t+1} - \mathbf{f}_t\,]\,]^{-1} \cdot \mathrm{Cov}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\tilde{R}^{fh}_{\mathcal{P},t+1}\,], \mathrm{E}_{\mathbb{Q}}[\,\mathbf{e}_{t+1} - \mathbf{f}_t]\,], \tag{20}$$

where matrix $\mathrm{Var}_{\mu}(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t])$ has to be positive definite in order its inverse exists. Similarly as how risk is captured in the reduced probability $\bar{\mathbb{Q}}$ for $\lambda \to \infty$ in Eq. (19), here we obtain an analogous result for $\theta \to \infty$, where the uncertainty is captured in μ, the agent's prior probability on the space of possible models. Since $\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t]$ is a random variable associated to each model $\mathbb{Q}$, this unpredictability is the only uncertainty that matters in the limiting case of $\theta \to \infty$. If the underlying models exhibit a negative correlation between the predictions of hedged portfolio returns and exchange rates, this implies a positive optimal ambiguity minimizing currency exposure given in (20), and the converse for the positive correlation. The intuition is the same as for the risk only minimizing case from above. The only difference is that, for $\theta \to \infty$, the ambiguity is completely captured in μ, compared to the risk, for $\lambda \to \infty$, being entirely captured in $\bar{\mathbb{Q}}$.

Mark that the general solution given in (18) allows for an arbitrary combination of purely risky and ambiguous[12] assets with the variance-covariance matrix $\mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t]\,]$ being positive semi-definite.

[12] Throughout the work we use the term ambiguous for assets who are risky and ambiguous simultaneously and purely risky for assets which are risky but not ambiguous.

However, in a limiting case (20), where $\theta \to \infty$, this only makes conceptual sense when all assets are ambiguous and their corresponding variance-covariance matrix is positive definite.[13]

Special Case: Ambiguity-Neutral Investor

Take an ambiguity neutral agent with $\theta \to 0$. This is equivalent to the mean–variance setting where all uncertainty is captured in $\bar{\mathbb{Q}}$ and the optimal currency exposure is obtained as

$$\begin{aligned}\boldsymbol{\Psi}^*_{t,mv} &= -\mathrm{Var}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1} - \mathbf{f}_t\,]^{-1} \cdot \Big(\mathrm{Cov}_{\bar{\mathbb{Q}}}[\,\tilde{R}^{fh}_{\mathcal{P},t+1}, \mathbf{e}_{t+1} - \mathbf{f}_t\,] - \frac{1}{\lambda}\mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1} - \mathbf{f}_t]\Big) = \\ &= \boldsymbol{\Psi}^*_{t,risk} + \frac{1}{\lambda}\mathrm{Var}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1} - \mathbf{f}_t\,]^{-1} \cdot \mathrm{E}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1} - \mathbf{f}_t\,].\end{aligned} \tag{21}$$

Notice that the optimal mean–variance currency exposure can be expressed as a sum of the optimal minimum variance exposure from (19) and the market price of currency risk adjusted for risk aversion λ. The market price of currency risk represents the trade-off between the expected excess return on currencies and the variance of these returns. Its effect on the optimal mean–variance currency exposure vanishes as $\lambda \to \infty$. Under the risk-neutral measure, a forward rate is an unbiased predictor of the future spot rate. However, in the real world measure, this does not hold in general and investor's predictive models aim to capture that. In the case of $\mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1} - \mathbf{f}_t] > 0$, the optimal mean–variance currency exposure increases for the risk aversion adjusted market price of currency risk. An agent increases her foreign currency exposure since she expects to get a higher risk adjusted return from the unhedged position compared to return from the hedged position. Since the additional risk is compensated, she becomes, in the mean–variance sense, better off. The converse is true for $\mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1} - \mathbf{f}_t] < 0$. In the case where $\mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1} - \mathbf{f}_t] = 0$, hedging does, in expectation, not affect the currency returns and the optimal mean–variance exposure equals the optimal exposure (19) affected by risk only. We can argue that starting with the infinitely risk averse agent from (19), Eq. (21) presents a generalization to the case where risk adjusted deviations from the uncovered interest parity are explicitly captured in the optimal currency exposure.

We can look at the optimal mean–variance currency exposure as an unambiguous currency overlay strategy. The focus of the minimum variance currency exposure is strictly to eliminate risk, no part of $\boldsymbol{\Psi}^*_{t,risk}$ seeks to add an additional return to the portfolio. The optimal mean–variance currency exposure then introduces an additional component seeking to add a source of excess return (commonly referred to as "alpha"). This is also a characteristic of a currency overlay strategy. An initial currency exposure $\boldsymbol{\Psi}^*_{t,risk}$ is determined and the portion of exposure on top of that is then seeking to generate alpha by exploiting (predicting) currency movements, which then leads to $\boldsymbol{\Psi}^*_{t,mv}$. Importantly, the decision of how to optimize the currency risk-return spectrum is independent of the decision of how to allocate a portion of the portfolio to different foreign denominated assets. It is also essential to

[13] This is conceptually similar to the non-existence of an inverse of a covariance matrix of an (excess) return vector with a riskless asset as at least one of the vector components.

understand that a currency overlay (optimal currency exposure) is not a direct investment. It is a risk management strategy, in our case implemented with forward contracts, that does not require any investment or additional asset allocation.

Rewriting the general version of optimal currency exposure given in (18), one can show that

$$\mathbf{\Psi}^*_t = \mathbf{\Psi}^*_{t,mv}+\big(\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,] + \theta \mathrm{Var}_\mu[\,\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t]\,]\big)^{-1}\,\theta \mathrm{Var}_\mu[\,\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t]\,]\cdot(\mathbf{\Psi}^*_{t,amb}-\mathbf{\Psi}^*_{t,mv}), \quad (22)$$

where the expression from (21) is used and the positive definiteness of $\mathrm{Var}_\mu[\,\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t]\,]$ is assumed in order to use (20). The detailed proof of (22) can be found in the Appendix. We can use similar reasoning as above and argue that (22) presents an explicit correction of the optimal currency exposure arising from a generalization of an unambiguous (purely risky) mean–variance agent to the decision maker with an aversion to ambiguity. The term $(\mathbf{\Psi}^*_{t,amb}-\mathbf{\Psi}^*_{t,mv})$ quantifies the deviation of optimal currency exposure for an infinitely ambiguity averse agent and an unambiguous agent. In case of the two being equal, there is no exposure correction compared to the unambiguous mean–variance agent, and the correction grows larger in the difference of the two. The term in front of $(\mathbf{\Psi}^*_{t,amb}-\mathbf{\Psi}^*_{t,mv})$ accounts for the size and direction of the model uncertainty correction and vanishes for $\theta \to 0$. Hence, it can be understood as the ambiguity correction equivalent to the market price of currency risk from above.

Corollary. *Here, we present a setup in which we differentiate between a purely risky and an ambiguous asset. This problem is in the ambiguity aversion literature known as a natural extension of the standard (riskless and risky asset) portfolio problem to the setting of model uncertainty. International portfolio allocation problems provide a natural application of this setting, with domestic government bonds viewed as risk-free, other domestic assets viewed as purely risky, and foreign assets and exchange rates viewed as ambiguous.*

Take an investor who is fully invested in a portfolio of domestic assets and is considering whether exposure to other currencies would help improving her portfolio risk-ambiguity-return spectrum. We treat the domestic portfolio position as purely risky and the foreign currency positions as ambiguous. This corresponds to $\mathrm{E}_{\mathbb{Q}}[\,\tilde{R}^{fh}_{\mathcal{P},t+1}\,]$ *being a constant for all models* $\mathbb{Q}$ *and implies* $\mathrm{Cov}_\mu[\,\mathrm{E}_{\mathbb{Q}}[\,\tilde{R}^{fh}_{\mathcal{P},t+1}\,], \mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t]\,] = 0$. *Using Eq. (18), the optimal currency exposure is obtained as*

$$\mathbf{\Psi}^*_{t,expl} = -\Big(\mathrm{Var}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,] + \frac{\theta}{\lambda}\mathrm{Var}_\mu[\,\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t]\,]\Big)^{-1}\cdot\Big(\mathrm{Cov}_{\bar{\mathbb{Q}}}[\,\tilde{R}^{fh}_{\mathcal{P},t+1},\mathbf{e}_{t+1}-\mathbf{f}_t\,] - \frac{1}{\lambda}\mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1}-\mathbf{f}_t]\Big). \quad (23)$$

Observe that in the limit when $\lambda \to \infty$, *the optimal currency exposure converges to the minimum variance case (infinitely risk averse agent) given in (19). Moreover, in the case when an agent is becoming infinitely ambiguity averse with* $\theta \to \infty$, *the optimal currency exposure converges to zero for all foreign currencies* $c = 2, 3, \ldots, M+1$ *and the entire currency exposure is kept solely in the domestic* *currency. Maggiori, Neiman, and Schreger (2018) demonstrate that investors exhibit home currency bias in that they disproportionately hold securities denominated in their domestic currency. In the example above, where we treated the domestic portfolio position as purely risky, we showed that large ambiguity towards the uncertain exchange rates can explain why investor holdings are biased toward their own currencies. This shows that the puzzle of insufficient currency diversification can also be driven by investor's ambiguity aversion.*

3.3.2 Sample Efficiency

The inputs to various types of portfolio optimization problems are usually estimated with large errors. We presented a setting which accounts for such model uncertainty in the expectation of future returns. This approach can be very useful for asset managers and other practitioners which deal with determining the future looking optimal currency exposure of their portfolio. On the other hand, researchers are also interested in the historical optimality and the role of sampling error in the construction of the ex-post (in-sample) efficient portfolio weights, or in our case currency exposures. For example, Britten-Jones (1999) showed that the ordinary least squares (OLS) regression of a constant onto a set of asset's excess returns, without an intercept term, results in an estimated coefficient vector that is a set of risky-asset-only portfolio weights for a sample efficient mean–variance portfolio. Similar idea applied to the optimal currency exposure is presented in Campbell, Serfaty-De Medeiros, and Viceira (2010). The authors derive an approximation to the ex-post minimum variance optimal currency exposure by regressing portfolio logarithmic excess returns on a constant and a vector of currency excess returns, and switching the sign of the slope. Let us analyze how can our setting, with the addition of model uncertainty, be compared to the approaches presented in the existing literature.

In what follows we work with the demeaned historical returns, which come from the sample measure $\mathbb{H}$. Define an empirical loss function $\mathcal{L}_{\mathbb{H}}(\tilde{R}^h_{\mathcal{P},t+1}) := -U(\tilde{R}^h_{\mathcal{P},t+1})$ as a negative of robust mean–variance utility function. This choice implies

$$-\min_{\mathbf{\Psi}_t} \mathcal{L}_{\mathbb{H}}(\tilde{R}^h_{\mathcal{P},t+1}) = \max_{\mathbf{\Psi}_t} U(\tilde{R}^h_{\mathcal{P},t+1}),$$

which shows the equivalence between minimizing the chosen loss function and maximizing utility. The argument which minimizes the empirical loss function is the optimal in-sample currency exposure.[14] It can be found as

$$\arg\min_{\mathbf{\Psi}_t} \mathcal{L}_{\mathbb{H}}(\tilde{R}^h_{\mathcal{P},t+1}) = \arg\min_{\mathbf{\Psi}_t} \left\{ \lambda \widehat{\mathrm{Var}}_{\mathbb{H}}[\, \tilde{R}^h_{\mathcal{P},t+1}\,] + \theta \mathrm{Var}_{\mu}[\, \mathrm{E}_{\mathbb{Q}}[\, \tilde{R}^h_{\mathcal{P},t+1}\,]\,] \right\}, \tag{24}$$

[14] Note the connection to empirical risk minimization from statistical learning theory, where one can only measure the performance of an algorithm on a known set of training data and does not know the true distribution of the underlying data.

where the expectation term from the utility function vanishes because of demeaning, $\widehat{\mathrm{Var}}_{\mathbb{H}}[\,\tilde{R}^{h}_{\mathcal{P},t+1}\,]$ is the maximum likelihood sample covariance matrix of hedged portfolio returns, and measures $\mathbb{Q}$ represent the model uncertainty perceived by an agent in the sample period.

Let $\mathbf{X}$ denote the $(T \times M)$ matrix of demeaned historical currency excess returns $\mathbf{e}_{t+1} - \mathbf{f}_t$, and let $\mathbf{y}$ denote the $(T \times 1)$ vector of demeaned historical fully hedged portfolio return $\tilde{R}^{fh}_{\mathcal{P},t+1}$, where T is the number of observations in the sample. Then, we have

$$\widehat{\mathrm{Var}}_{\mathbb{H}}[\,\mathbf{e}_{t+1} - \mathbf{f}_t\,] = \frac{1}{T}\mathbf{X}'\,\mathbf{X}$$
$$\widehat{\mathrm{Cov}}_{\mathbb{H}}[\,\tilde{R}^{fh}_{\mathcal{P},t+1}, \mathbf{e}_{t+1} - \mathbf{f}_t\,] = \frac{1}{T}\mathbf{X}'\,\mathbf{y},$$

by employing the (maximum likelihood) sample covariance matrix. Let $\mathbf{W} = \frac{\lambda}{T}\mathbf{I}$, where $\mathbf{I}$ is a $(T \times T)$ identity matrix, $\mathbf{Z} = \theta \mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\tilde{R}^{h}_{\mathcal{P},t+1}]\,]$ and $\mathbf{z_0} = -\boldsymbol{\Psi}^{*}_{t,amb}$, where we assumed that the inverse of $\mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t]\,]$ exists. We can express the loss function as

$$\mathcal{L}_{\mathbb{H}}(\tilde{R}^{h}_{\mathcal{P},t+1}) = \boldsymbol{\Psi}_t{}'\,\mathbf{X}'\,\mathbf{W}\mathbf{X}\boldsymbol{\Psi}_t + 2\boldsymbol{\Psi}_t{}'\,\mathbf{X}'\,\mathbf{W}\mathbf{y} + \boldsymbol{\Psi}_t{}'\,\mathbf{Z}\boldsymbol{\Psi}_t + 2\boldsymbol{\Psi}_t{}'\,\mathbf{Z}\mathbf{z_0} + \mathrm{rest}, \tag{25}$$

where we explicitly write the terms which depend on $\boldsymbol{\Psi}_t$ and with rest denote other terms which do not affect the optimization. Using Eq. (25), the minimization can be rewritten as

$$\begin{aligned} \arg\min_{\boldsymbol{\Psi}_t} \mathcal{L}_{\mathbb{H}}(\tilde{R}^{h}_{\mathcal{P},t+1}) &= \arg\min_{\boldsymbol{\Psi}_t} \left\{ \Big(\mathbf{y} + \mathbf{X}\boldsymbol{\Psi}_t\Big)'\,\mathbf{W}\Big(\mathbf{y} + \mathbf{X}\boldsymbol{\Psi}_t\Big) + (\boldsymbol{\Psi}_t + \mathbf{z_0})'\,\mathbf{Z}(\boldsymbol{\Psi}_t + \mathbf{z_0}) + \mathrm{rest} \right\} = \\ &= \arg\min_{\boldsymbol{\Psi}_t} \big\|\mathbf{y} - \mathbf{X}(-\boldsymbol{\Psi}_t)\big\|^2_{\mathbf{W}} + \big\|(-\boldsymbol{\Psi}_t) - (-\boldsymbol{\Psi}^{*}_{t,amb})\big\|^2_{\mathbf{Z}}, \end{aligned} \tag{26}$$

where $\big\|\boldsymbol{\Psi}_t\big\|^2_{\mathbf{D}} = \boldsymbol{\Psi}'_t\,\mathbf{D}\boldsymbol{\Psi}_t$ stands for the weighted L^2-norm given a positive definite matrix $\mathbf{D}$.

Corollary. *Eq.* (26) *shows that the optimal ex-post currency exposure for a risk-and-ambiguity-averse agent can be found as a generalized ridge regression of the demeaned historical hedged portfolio returns on the demeaned currency excess returns and shrunk towards the infinitely ambiguity averse optimal exposure distorted by the level of uncertainty. A generalized ridge regression estimator minimizes a weighted least squares augmented with a generalized ridge penalty. The weighting matrix is given by* $\mathbf{W} = \frac{\lambda}{T}\mathbf{I}$ *and accounts for agent's risk aversion* λ*. The generalized penalty is given by* $\mathbf{Z} = \theta \mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\tilde{R}^{h}_{\mathcal{P},t+1}\,]\,]$ *and accounts for model uncertainty and ambiguity aversion* θ*. The shrinkage target is given by* $-\boldsymbol{\Psi}^{*}_{t,amb}$*, which is a negative of optimal currency exposure for an infinitely ambiguity averse agent* (20)*. The optimal ex-post currency exposure minimizing the loss function given in Eq.* (24) *is then obtained as a negative of the described generalized ridge regression estimator. We just showed that above stated in-sample loss function minimization problem can be solved using an artificial generalized ridge regression which recovers the sample efficient currency exposure for a risk-and-ambiguity-averse investor.*

In the limiting case of no model uncertainty with $\theta \to 0$, the penalty term vanishes. Optimal risk reductive ex-post currency exposure can then be estimated via an OLS regression $\arg\min_{\mathbf{\Psi}_t} \left\|\mathbf{y} - \mathbf{X}(-\mathbf{\Psi}_t)\right\|_{\mathbf{I}}^2$ of demeaned hedged portfolio returns on the demeaned currency excess returns (in the usual L^2-norm) and obtained by changing the sign of the estimator. This is in line with the existing literature (e.g., see Campbell, Serfaty-De Medeiros, and Viceira, 2010; Schmittmann, 2010, and others) where the vector of optimal currency exposure is obtained as the negative of the slopes (without an intercept) of a multiple regression of the excess portfolio return on a constant and the vector of currency excess returns. Notice that this is equivalent to demeaning the returns and regressing without a constant term. Hence, our paper nests the OLS approaches presented in the existing literature as a special case of no model uncertainty.

Including a constant would in our approach represent an unambiguous prospect and the corresponding generalized penalty matrix $\mathbf{Z}$ would not have been positive definite. It would not generate a proper norm, as well as the shrinkage target would not exist. Moreover, as presented in Hastie, Tibshirani, and Friedman (2009), an intercept term is always left out of the ridge penalty term. Penalization of the intercept would make the procedure depend on the origin chosen for the predictor variable. This is the reason our approach uses the demeaned returns and regresses without the intercept term. Note that the lack of an intercept implies that residuals need not sum to zero.

Moreover, note that in addition to centralization of the variables, we did not impose the scaling of the independent variables such that they have variance equal to 1. The reason for standardizing is that the ridge penalty term is applied uniformly across the predictors, and including variables with different scales unevenly penalizes the coefficients. This is especially important when the independent variables are not given in the same unit (but for example as centimeters and kilograms). In such case, the shrinking becomes discriminatory. However, in our setting, all of the regressors are given as (excess) returns and consequently yield comparable contribution to the penalization. The roles of the variables are equivalent which leads to no artificially inflated values arising due to no scaling. This is the reason that centering (demeaning) of the variables is sufficient in our setting.

Lastly, note that the reason for the possibility of the in-sample interpretation of optimal currency exposure as regression coefficients arises from the linear relationship between the portfolio return and currency exposure (zero value currency portfolio) and the choice of (robust) mean–variance preferences. The linearity is inherited since forward contract is an instrument with a linear pay-off. Moreover, ambiguity is measured in a quadratic manner, which enables that the solution is obtained by minimizing the weighted L^2-norm (generalized ridge regression). Hence, the differentiability and smoothness are preserved and the explicit solution exists.

3.3.3 Geometric Interpretation

The regression in (26) is slightly unusual, there is no intercept and the residual vector can be correlated with the regressors. It can be regarded as an artificial regression which recovers the sample efficient currency exposure and at the same time provides a useful geometric interpretation.

Let us start with a geometric interpretation of the risk only version which corresponds to the ordinary least squares linear regression. The dependent variable $\mathbf{y}$ represents the return on a fully hedged portfolio that has no exposure to currency risk, the independent variables $\mathbf{X}$ involve only the excess returns on currencies, coefficients $\mathbf{\Psi}_t$ represent the currency exposures (weights of a zero value currency portfolio), and the regression residuals show the deviation of the pure currency portfolio $\mathbf{X\Psi}_t$ from the fully hedged portfolio returns $\mathbf{y}$. The estimated optimal in-sample currency weights produce a pure currency exposure which is closest in terms of least squares distance to the fully hedged portfolio returns. The closer the two, the larger amount of risk can be reduced via the currency position, where the optimal risk reductive currency exposure corresponds to the negative of the estimated regression coefficient $-\mathbf{\Psi}_t$. In such way a sample efficient risk reduction is obtained.

The situation gets slightly more complicated in the presence of model uncertainty. The generalized penalty term corresponds to the utility loss arising from model uncertainty. The ordinary least squares regression presents a geometric interpretation as an orthogonal projection of hedged portfolio returns onto the space spanned by the currency excess returns. In the case of generalized ridge regression, the resulting coefficients (optimal exposures) still lie in the span of predictors (excess currency returns), whereas the shrinkage (of the coordinates obtained via OLS) is induced. The magnitude of penalizing factor θ and the structure of model uncertainty $\mathrm{Var}_\mu[\,\mathrm{E}_\mathbb{Q}[\,\tilde{R}^h_{\mathcal{P},t+1}\,]\,]$ control the distance and direction of shrinkage towards the target $-\mathbf{\Psi}^*_{t,amb}$. The generalized penalty is a quadratic form. It geometrically implies a non-zero centered, ellipsoid parameter constraint. This can be seen in Figure 1, where the red point corresponds to the two-dimensional vector of optimal currency exposure obtained by the weighted least squares regression, without regularization. The blue point represents the shrinkage target and the black ellipsoid contains all sets of currency exposures that are attainable. For larger penalty term θ this set becomes smaller since the tighter constraint binds, and the converse for the smaller penalty. The shape of the ellipsoid is determined by the model uncertainty matrix $\mathrm{Var}_\mu[\,\mathrm{E}_\mathbb{Q}[\,\tilde{R}^h_{\mathcal{P},t+1}\,]\,]$. The optimal solution is given as the tangent of the weighted least squares residual sums of squares contour sets and the set of attainable solutions. Note that in the classical ridge regression the penalty matrix is an identity multiplied with the scalar penalty and the target is set at the origin. This implies a sphere at the origin representing the attainable set of points. In the generalized ridge regression this sphere becomes an ellipsoid centered at the particular shrinkage target.

The weighted least squares regression sum of squares are convex in $\mathbf{\Psi}_t$, as well as the ridge penalty is convex in $\mathbf{\Psi}_t$. This implies a unique minimizer of the penalized sum of squares. The explicit solution

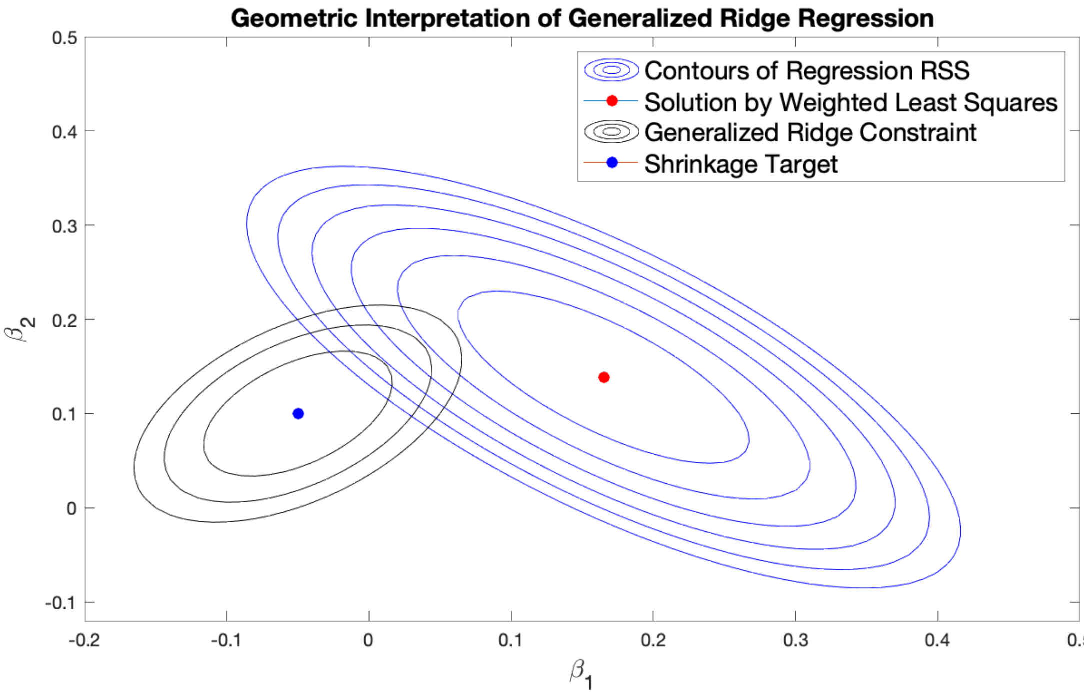

Figure 1 – This figure presents a geometric interpretation of a generalized ridge regression in two dimensions. The red point corresponds to the solution obtained by weighted least squares. Around the red point the contours of the regression residual sums of squares are presented. The blue point is the shrinkage target and the ellipsoid around that point represents the contours of attainable points (given particular penalty parameters). The set of attainable points shrinks towards the target for a larger penalization parameter. The shape of the ellipsoid is determined by the positive definite matrix in the weighted L^2-norm. The optimal solution of this optimization problem is given as the tangent of the weighted least squares contour sets and the set of attainable points.

of Eq. (26) yields a sample efficient optimal risk and ambiguity currency exposure and is given by

$$\begin{aligned}\widehat{\mathbf{\Psi}}^*_{t,\mathbb{H}} &= -\left(\mathbf{X}'\mathbf{W}\mathbf{X}+\mathbf{Z}\right)^{-1}\left(\mathbf{X}'\mathbf{W}\mathbf{y}+\mathbf{Z}\mathbf{z_0}\right) = \\ &= \widehat{\mathbf{\Psi}}^*_{t,\mathbb{H},risk} + \left(\mathbf{X}'\mathbf{W}\mathbf{X}+\mathbf{Z}\right)^{-1}\mathbf{Z}\left(\mathbf{\Psi}^*_{t,amb} - \widehat{\mathbf{\Psi}}^*_{t,\mathbb{H},risk}\right),\end{aligned} \tag{27}$$

where $\widehat{\mathbf{\Psi}}^*_{t,\mathbb{H},risk} = -(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$ is the sample efficient risk only minimizing currency exposure obtained by ordinary least squares. The proof of (27) is given in the Appendix. Observe that this equation represents the in-sample equivalent of the general future looking solution given in (18) and (22). It also provides an intuition for how model uncertainty affects optimal currency exposure. The correction term arising from model uncertainty is affected by the difference between $\mathbf{\Psi}^*_{t,amb}$ and $\widehat{\mathbf{\Psi}}^*_{t,\mathbb{H},risk}$ transformed with $\left(\mathbf{X}'\mathbf{W}\mathbf{X}+\mathbf{Z}\right)^{-1}\mathbf{Z}$ which vanishes for $\theta \to 0$. In the particular case of $\mathbf{\Psi}^*_{t,amb} = \widehat{\mathbf{\Psi}}^*_{t,\mathbb{H},risk}$, the shrinkage target matches the ordinary least squares solution (this geometrically corresponds to the red and blue point matching). In this case, the risk only sample efficient solution is attainable for any θ and the model uncertainty does not affect the solution obtained by ordinary least squares.

Observe that the penalty term from Eq. (26) represents the utility loss from model uncertainty in the given sample period. One could think of risk minimizing ex-post efficient currency exposure $\widehat{\mathbf{\Psi}}^*_{t,\mathbb{H},risk}$

as the first-best in the case of no model uncertainty. Then, in the presence of model uncertainty, $\widehat{\boldsymbol{\Psi}}^*_{t,\mathbb{H}}$ provides the solution to the second-best, taking into account the structure of agent's uncertainty and ambiguity aversion θ. This correction of optimal currency exposure from the first-best to the second-best is geometrically represented in Figure 1. Note that the optimal currency exposures (weights of a zero investment currency portfolio) are still acquired in the space spanned by currency excess returns and the deviation (shrinkage) from an orthogonal projection is determined by the ridge regularization penalty term.

Example. *Allow us to revisit the example of an agent fully invested in a portfolio of domestic assets who considers other currencies in order to improve her risk-ambiguity-return spectrum. The future looking (out-of-sample) solution is given in Eq. (23). Treating the domestic position as purely risky and foreign currency positions as ambiguous implies* $\mathrm{Cov}_\mu[\,\mathrm{E}_\mathbb{Q}[\,\tilde{R}^{fh}_{\mathcal{P},t+1}\,], \mathrm{E}_\mathbb{Q}[\,\mathbf{e}_{t+1} - \mathbf{f}_t\,]) = 0$ *and hence* $\boldsymbol{\Psi}^*_{t,amb} = 0$. *This geometrically corresponds to the shrinkage target based in the origin. Taking* $\theta \to \infty$ *shrinks the set of attainable currency exposures to a single point in the origin. Since the structure of model uncertainty did not change, this is in line with the result obtained for the future looking case. Such infinite penalization represents a large under-fitting (in a regularization framework) and corresponds to the largest deviation from the first-best to the second-best in the sense of utility loss and optimal exposure weights, given a particular shrinkage target.*

The optimal in-sample exposure estimate can also be found by using the sample estimators of the variance-covariance matrix under $\mathbb{H}$ and utilizing Eq. (18). However, the regression approach provides a simple tool for the empirical analysis which allows for inference procedures for hypotheses about the efficient currency allocation. The use of these inference procedures shows the importance and magnitude of sampling error in estimates of the efficient currency exposure of an international portfolio. Moreover, the regression framework highlights the stochastic nature of the variables estimated from sample data. The regularization part (model uncertainty) induces a bias-variance trade-off of the estimated sample efficient currency allocation. The estimates become biased, whereas obtained confidence intervals are shrunk, given the structure of model uncertainty which controls the distance and direction of shrinkage towards the infinitely ambiguity averse target. A large aversion to ambiguity could then also be interpreted as an in-sample under-fitting, which makes the distribution of optimal currency exposure too regular and thus largely biased.

We showed that there is a direct link, an equivalence, between a robust mean–variance utility representation and the solution to such utility maximization problem obtained with a generalized ridge regression. In such way we formally associated the areas of financial economics (asset/currency allocation) with statistical learning (regularization). An interesting question arises, namely, if there exists a class of utility functions which would yield an equivalent representation via a lasso regression (penalization using L^1-norm). Ridge regression corresponds to smooth shrinkage where differentiability is preserved and the solution is obtained in a closed form. On the other hand, lasso regression corresponds to an extreme shrinkage (sparsity), penalization is performed in a non-differentiable L^1-norm and the solution has to be obtained numerically.[15] In the work of Brodie, Daubechies, De Mol, Giannone, and Loris (2009), the authors show that adding a lasso penalty to the optimization problem corresponds to accounting for transaction costs. This gives rise to a suggestion that an elastic net regularization, a regression method that linearly combines the L^1 and L^2 penalties of the lasso and ridge methods, could be a framework that captures both transaction costs and model uncertainty.

4 Empirical Analysis

We turn our attention now to the empirical analysis of the results derived above. First, we conduct an in-sample analysis of the effects of ambiguity on the optimal currency exposure. Subsequently, we provide an out-of-sample back test of the designed hedging strategy.

The empirical study presented in this paper encompasses seven major developed economies: Australia, Canada, Switzerland, Eurozone, United Kingdom, Japan and United States. We have collected daily time series series of spot and forward exchange rates, short-term interest rates (up to the tenor of one year), broad equity market total return indices, and government bonds total return indices (for various maturity buckets) obtained from Thomson Reuters Datastream and Bloomberg. Our sample straddles the period from January 1999, i.e., since the dawn of the euro, until June 2018. Using recent market data enables us to detect possible shifts of the optimal currency exposures which occurred during or after the major financial and economic crises, e.g., the Dot-Com Bubble 2000–2002, the Global Financial Crisis 2008–2009, and the Euro Sovereign Debt Crisis 2009–2011. We study the impact of model uncertainty and compare our empirical results to risk-only approaches demonstrated in the existing literature.

4.1 In-Sample Analysis: The Impact of Ambiguity Aversion

In the theoretical part of this paper we have introduced the optimal currency exposure problem for an agent who exhibits robust mean–variance preferences and differentiates between purely risky and ambiguous assets. Following that suit, we consider a fully hedged portfolio as a purely risky investment, and a portfolio exposed to foreign currency fluctuations as an ambiguous investment.[16] As our focus in this section is entirely geared towards sample estimates, we employ again the notation of historical measure $\mathbb{H}$. We further assume that the uncovered interest rate parity holds, thus the expected returns drop out from our calculations. Arguably, the sample estimates of expected returns are very noisy and

[15] One can make similar arguments also for the cases where penalization is performed via L^0- or L^∞-norms.

[16] The economic interpretation of a fully hedged portfolio as a purely risky investment means, from the mathematical point of view, that the ridge regression target is centered at the origin.

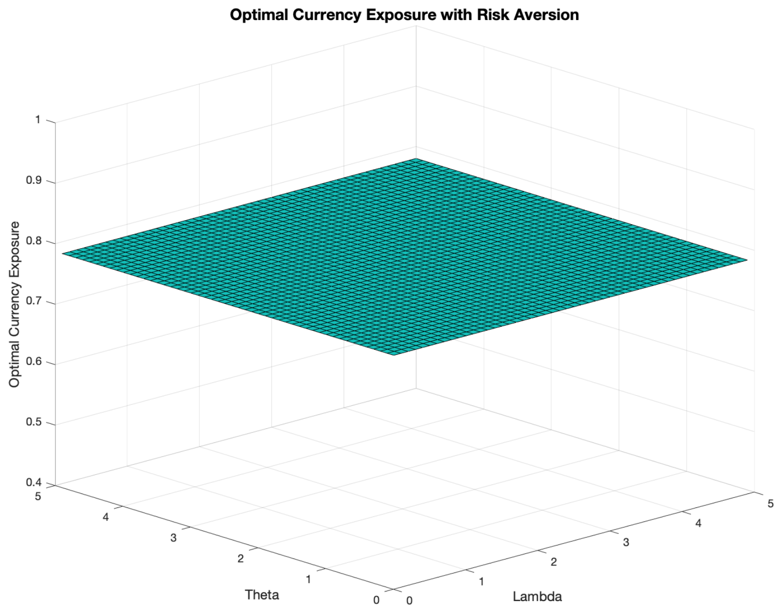

Figure 2 – Optimal currency exposure in CHF (for a EUR based investor) in dependence of risk and ambiguity aversion parameters λ and θ is plotted here. We assume no ambiguity $\mathrm{Var}_\mu[\,\mathrm{E}_\mathbb{Q}[e_{t+1} - f_t]\,] = 0$ and uncovered interest rate parity to hold. The optimal currency exposure hence corresponds to the minimum variance case and does not depend on λ or θ.

can significantly limit the informational content of our in-sample analysis. Consequently, our model setting represents the reduced form of Eq. (23):

$$\widehat{\mathbf{\Psi}}^*_{t,\mathbb{H},expl} = -\Big(\widehat{\mathrm{Var}}_\mathbb{H}[\,\mathbf{e}_{t+1} - \mathbf{f}_t\,] + \frac{\theta}{\lambda}\mathrm{Var}_\mu[\,\mathrm{E}_\mathbb{Q}[\mathbf{e}_{t+1} - \mathbf{f}_t]\,]\Big)^{-1} \cdot \Big(\widehat{\mathrm{Cov}}_\mathbb{H}[\,\tilde{R}^{fh}_{\mathcal{P},t+1}, \mathbf{e}_{t+1} - \mathbf{f}_t\,]\Big). \tag{28}$$

To simplify our analysis and provide intuition about the effects of ambiguity aversion on optimal currency exposure, we assume independence of agent's estimates with $\mathrm{Var}_\mu[\,\mathrm{E}_\mathbb{Q}[\,\mathbf{e}_{t+1} - \mathbf{f}_t\,]\,] = \frac{1}{T^2}\mathbf{I}$. Here, T is the number of observations in the sample and $\mathbf{I}$ is the identity matrix, scaled in order to be of comparable order of magnitude. This assumption allows us to perform an in-sample analysis without explicitly specifying various predictive models. We stress that the underlying assumptions transform our setting to the ordinary (non-generalized) ridge regression.[17]

Let us start with an analysis of optimal currency exposure in dependence of risk and ambiguity aversion parameters λ and θ. Notice that in order to gain the intuition about the effects of risk and ambiguity aversion parameters in a three-dimensional plot, we have to work with the univariate case of optimal currency exposure (an exposure to a single foreign currency). Similar reasoning extends to the general multivariate case of optimal currency exposure, whereas the corresponding plots then become impossible to present.

[17] Using the notation from previous section, the expression in (28) can be rewritten as $\widehat{\mathbf{\Psi}}^*_{t,\mathbb{H},expl} = -(\mathbf{X}'\mathbf{X} + q\mathbf{I})^{-1}\mathbf{X}'\mathbf{y}$, where the ridge penalization parameter equals to $q = \frac{\theta}{\lambda T}$.

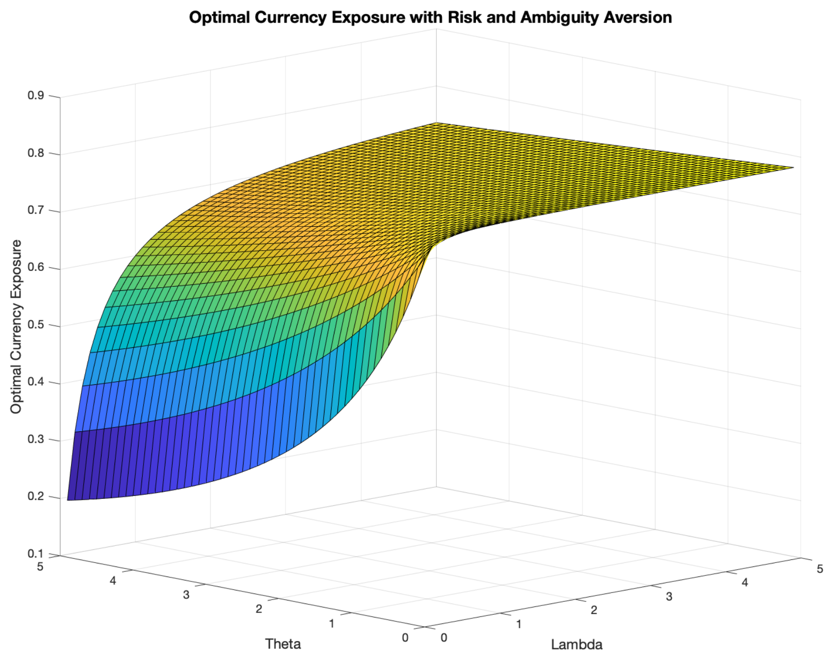

Figure 3 – Optimal currency exposure in CHF (for a EUR based investor) in dependence of risk and ambiguity aversion parameters λ and θ is plotted here. We assume independent prediction models with $\mathrm{Var}_\mu[\,\mathrm{E}_\mathbb{Q}[\,\mathbf{e}_{t+1} - \mathbf{f}_t\,]\,] = \frac{1}{T^2}\mathbf{I}$ and the uncovered interest rate parity to hold. When $\lambda \to \infty$, the optimal currency exposure converges to the minimum variance case and when $\theta \to \infty$, the optimal currency exposure converges to 0 (full hedging).

In Figure 2 observe the optimal Swiss franc exposure for a euro based investor, calculated under the assumption of no model uncertainty $\mathrm{Var}_\mu[\,\mathrm{E}_\mathbb{Q}[e_{t+1} - f_t]\,] = 0$, which corresponds to the current state of the literature. The optimal currency exposure is found via the ordinary least squares regression (minimum variance case) and is a constant function which does not depend on the choices of λ and θ.

The optimal currency exposure of the risk-and-ambiguity-averse agent is plotted in Figure 3. Observe the dependence of optimal Swiss franc exposure for a euro based investor, on the risk and ambiguity aversion parameters λ and θ. When $\lambda \to \infty$, the optimal currency exposure converges to the minimum variance case from Figure 2. However, when $\theta \to \infty$, the optimal currency exposure converges to 0 (full hedging), where the convergence for the lower values of λ is faster compared to the convergence for higher values of λ. As an agent is becoming more uncertain, for example, she does not trust her predictive models, her optimal choice is to hold less exposure to such ambiguous currencies.

One has to note that the optimal (in-sample) currency exposure is estimated from a finite sample. Being a function of the data, the optimal exposure estimator is itself a random variable, whose properties can be studied further. Hence, we analyze how different values of λ and θ affect the confidence intervals of historical optimal currency exposure. We form a monthly non-overlapping returns data set and assume these observations arise from an independent and identically distributed population. This allows us to use non-parametric bootstrapping (random sampling with replacement) in order to infer

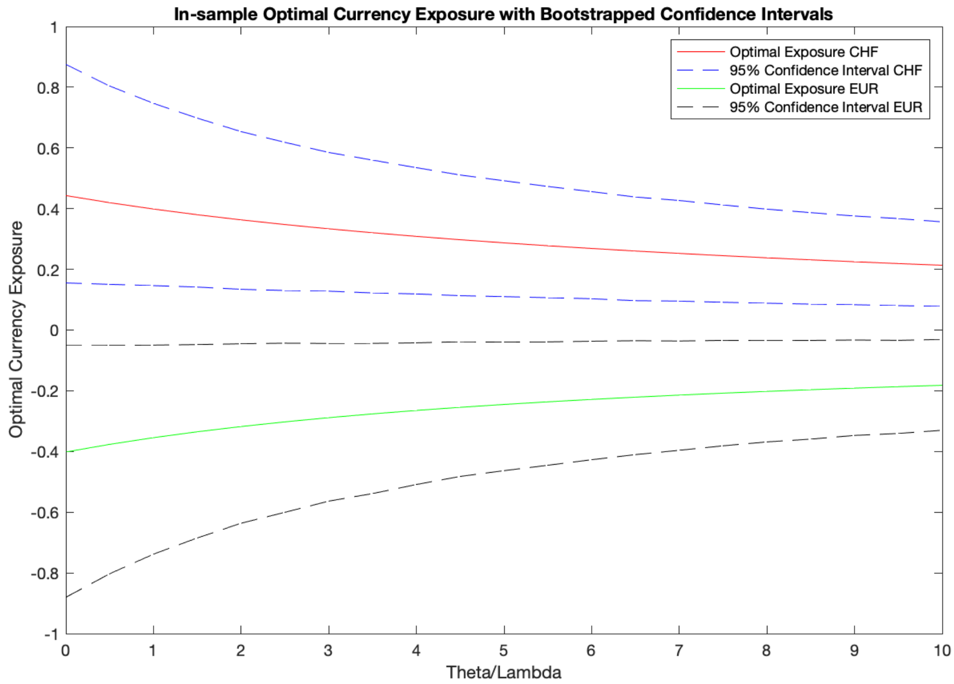

Figure 4 – Optimal currency exposure and the corresponding 95% confidence intervals for CHF and EUR (for a USD based investor) in dependence of risk and ambiguity aversion parameters λ and θ are plotted here. We assume independent prediction models with $\mathrm{Var}_\mu[\mathrm{E}_\mathbb{Q}[\mathbf{e}_{t+1} - \mathbf{f}_t]] = \frac{1}{T^2}\mathbf{I}$ and the uncovered interest rate parity to hold. Observe that as the bias increases for larger values of θ, simultaneously, the confidence intervals shrink (variance of the estimator decreases). This plot can also be interpreted as a ridge regularization path.

the confidence intervals of the sample estimate of optimal currency exposure.

We consider a portfolio based in US dollars with equally weighted exposures to Canadian, Swiss, Eurozone and USA equity and bond markets. Figure 4 depicts the estimated optimal exposure in Swiss francs and euros, together with the bootstrapped 95% confidence intervals for different choices of λ and θ. As the ambiguity aversion parameter corresponds to the penalization parameter of ridge regression, we observe an increase in bias for larger values of θ. Simultaneously, the confidence intervals shrink, representing the decrease in the variance of the optimal exposure estimator. In the limiting case of $\theta \to \infty$, the estimator converges to 0 (given our underlying economic assumptions), which leads to the largest achievable in-sample bias with zero variance. Note that this plot can also be interpreted as a ridge regularization path for exposures in CHF and EUR.

Moreover, the integration of the Swiss and European economies result in generally large positive correlations between the euro and Swiss franc. Thus, the optimal exposure (regression coefficient) in each of the two currencies can become poorly determined and exhibit high variance. One can observe this in the wide confidence intervals for $\theta = 0$ in Figure 4. Moreover, a large positive coefficient on Swiss franc is canceled by a similarly large negative coefficient on its correlated cousin euro. By imposing a size constraint on the coefficients, such as in ridge regression, this problem is alleviated. Hence,

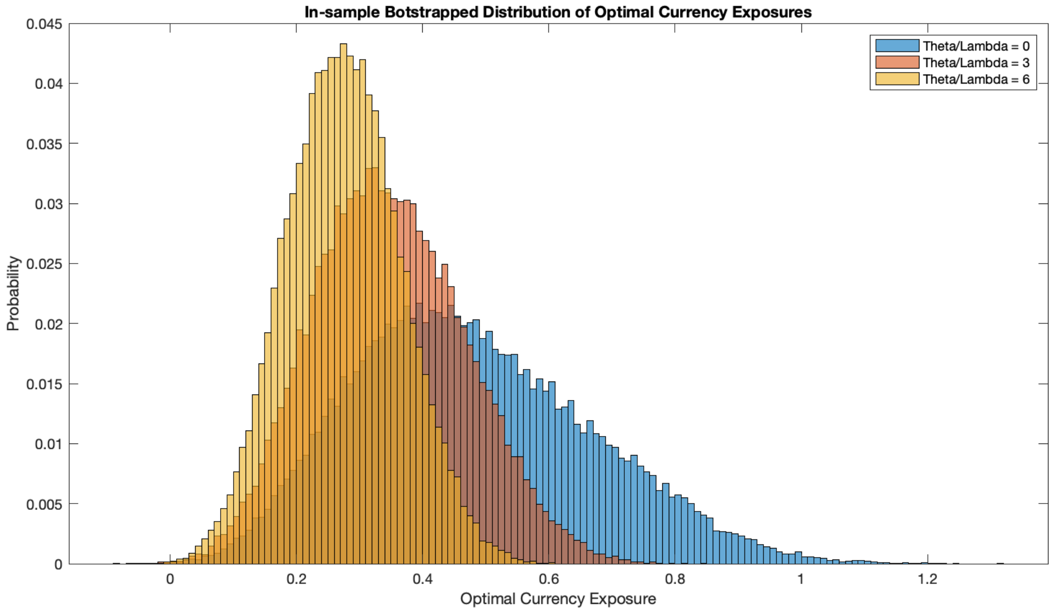

Figure 5 – Bootstrapped distribution of optimal currency exposure in CHF (for a USD based investor) for different values of risk and ambiguity aversion parameters λ and θ is plotted here. We assume independent prediction models with $\mathrm{Var}_\mu[\,\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t]\,] = \frac{1}{T^2}\mathbf{I}$ and the uncovered interest rate parity to hold. Observe the shift in bootstrapped distribution for larger values of θ.

ridge regression is a convenient alternative to OLS regression in the presence of correlated regressors.[18] This can be seen in an extensive narrowing of the confidence intervals for larger values of the ambiguity aversion parameter θ. Such narrowing is especially pronounced in the case of highly correlated variables, where utilizing ridge regression notably improves the estimation process.

The same phenomenon can be observed in Figure 5, where we plot the bootstrapped distribution of optimal Swiss franc exposure. The distribution for $\theta = 0$, which represents the current approaches in the literature (risk only), is extremely wide, showing that the estimator of optimal currency exposure exhibits large parameter uncertainty. For larger values of θ its distribution shifts towards the ridge target (which is equal to zero in our example), and narrows extensively. In the limit when $\theta \to \infty$, the distribution converges to the Dirac measure centered at the point 0.

Finally, in Figure 6 we present an efficient surface, which is a generalization of the standard efficient frontier to a three-dimensional setting. It is obtained as a maximization of expected portfolio return with respect to portfolio variance arising from risk as well as variance arising from ambiguity. Hence, it can be understood as a linear program with quadratic constraints.

In Figure 6 we consider an equally weighted global equity and bond portfolio for a USD based investor. Note that this surface is a result of choosing optimal currency positions while the portfolio assets are kept fixed. In order to obtain a finite frontier we have to allow for a finite amount of possible

[18] Ridge regression projects the predictor variable onto the principal components of input data points, and then shrinks the coefficients of the low–variance components more than the high-variance components. For more information, an interested reader can consult Hastie, Tibshirani, and Friedman (2009).

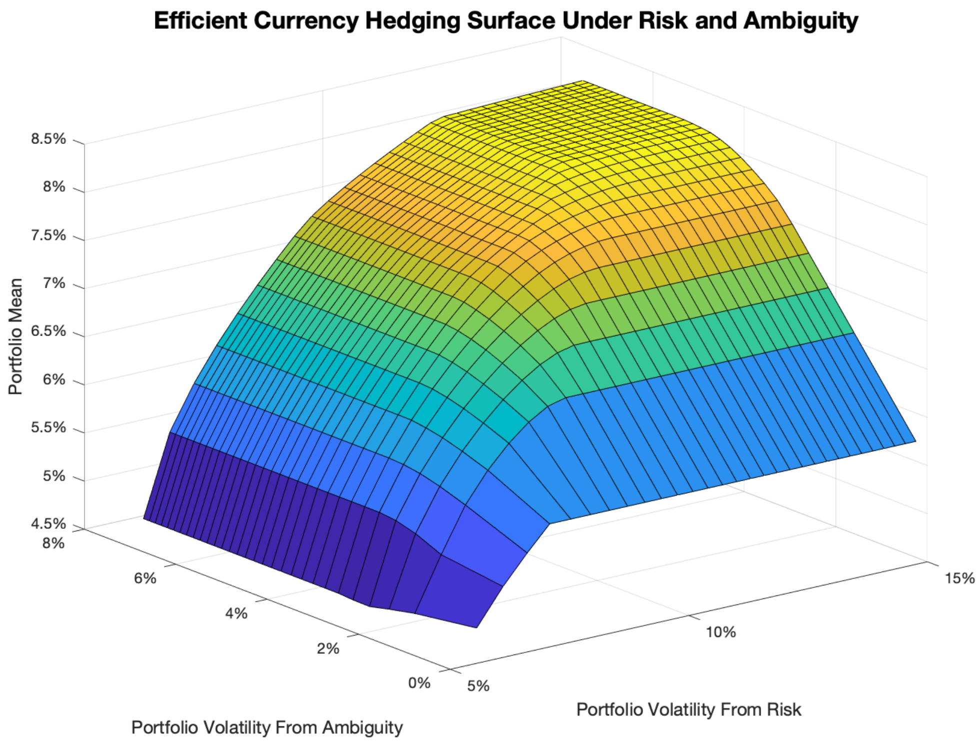

Figure 6 – Efficient surface in dependence of risk and ambiguity obtainable through changes in currency exposure is plotted here. We consider a global equally weighted equity and bond portfolio for a USD based investor. We assume independent prediction models with $\mathrm{Var}_\mu[\,\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t]\,] = \frac{1}{T^2}\mathbf{I}$. Observe that for a fixed amount of accepted ambiguity one obtains an ordinary efficient frontier.

leverage on currency positions. In this plot the maximal allowed leverage is equal to 100% of the current implicit currency position (in a long or short direction). For a larger allowed leveraged position the bounds of the efficient currency hedging surface increase. Without any constraint on currency positions, the problem becomes unbounded as infinite leverage is possible.

Observe that for a fixed number of accepted ambiguity, one obtains an ordinary efficient frontier. Moreover, the point corresponding to the minimum variance and maximum ambiguity (left bottom on the plot) converges to the global minimum variance case, which is typically the case considered in the existing literature. The convergence is achieved when the allowed leverage constrains increase enough that such solution is attainable.

4.2 Out-of-Sample Back Test

Above, we studied the effect of model uncertainty on the estimation of optimal currency exposure. The question that arises now is the performance of this theoretical model out-of-sample. In order to answer this question, we consider a global equally weighted mixed portfolio consisting of stocks, bonds and cash, and perform an out-of-sample back test for each of the seven base currencies. Our analysis uses the benchmarks of constant hedging (zero, half and full hedge), and compares them to different currency overlay strategies which we presented in the paper: the minimum risk (without model uncertainty), bounded minimum risk, minimum risk and ambiguity (robust setting), and bounded minimum risk and ambiguity hedging strategy. Unbounded models correspond to the OLS and ridge regressions, whereas the bounded models are solved via quadratic programming. In the bounded models we constrain the possible leverage to an excess of maximum 100% of the current implicit currency position (in long or short direction).

We consider a hedge maturity of 1 month.[19] When a currency forward contract used for hedging expires, its payoff is kept as domestic cash. Immediately, a new hedge is formed and the process is repeated over time. Note that we do not rebalance the asset positions, they stay constant over the whole back test. Their relative portfolio weights change over time, but not the absolute weights[20] which are set in the beginning of the back test such that the portfolio is equally weighted. The only changing variables are the positions in currency forward contracts (as determined by different models or kept constant in the case of constant hedging) and the domestic cash position (which depends on the currency forward pay-offs). We apply such procedure since our aim is to test the performance of different currency overlay strategies without any other effects on portfolio performance. Moreover, all results presented in this section are net of transaction (hedging) costs, which are assumed to be 5 basis points relative to the notional of the currency forward contract.

In Table 1, we report the annualized volatilities and Sharpe ratios of the portfolio daily returns, the average (monthly) hedging turnover, and the maximum drawdown of the portfolio, for each base currency and different hedging strategies described above, in comparison to the constant hedging benchmarks. The results in Table 1 show that the out-of-sample benefit of currency hedging depends sensitively on an investor's base currency as well as the model used for the hedging decision.

Full hedging decreases volatility for all base currencies. The decrease is substantial especially for Swiss and Japanese investors. These investors have a risk-reducing base currency, so they can reduce volatility by hedging back to that currency and out of foreign currencies. On the opposite, the volatility reduction from full currency hedging is particularly small for Australian and Canadian investors. All of the models outperform any type of constant hedging in the sense that they manage to additionally reduce volatility of portfolios. This reduction is the largest for Swiss investors, where the reduction of volatility is even around 50% compared to no hedging.

An important concern we investigate next is whether the volatility reductions come at the cost of lower realized average return (corrected for transaction costs) per unit of portfolio risk. Observe that the computed Sharpe ratios for constant hedging strategies usually perform fairly similarly across different base currencies. Notable exceptions are Australian and Swiss investors, for who full hedging notably increases the realized Sharpe ratios. Again, all of the models manage to perform better than constant hedging strategies. This happens across all base currencies and after correcting for transaction costs.

[19] Note that the results are robust also to alternative specifications.

[20] These are the weights denoted by $\lambda_{i,t}$ in Eq. (1).

Global Mixed (Equity, Bonds and Cash) Portfolio Out-of-sample Back Test Results

Base	Australia	Canada	Switzerland	Eurozone	UK	Japan	USA
Volatility							
No hedge	8.60%	9.53%	12.91%	10.22%	10.65%	16.44%	13.04%
Half hedge	6.90%	8.60%	10.20%	8.94%	9.50%	13.41%	11.58%
Full hedge	7.19%	8.94%	8.32%	8.29%	9.48%	11.95%	11.16%
Min Risk	5.77%	7.25%	6.81%	6.69%	7.49%	10.73%	9.14%
Bounded Min Risk	5.68%	7.08%	6.55%	6.63%	7.30%	9.29%	8.62%
Min Risk-Amb	5.68%	7.01%	6.53%	6.43%	7.22%	9.31%	8.66%
Bounded Min Risk-Amb	5.76%	7.16%	6.62%	6.59%	7.36%	9.33%	8.70%
Sharpe Ratio							
No hedge	0.55	0.64	0.30	0.50	0.69	0.43	0.56
Half hedge	0.72	0.65	0.35	0.53	0.69	0.42	0.56
Full hedge	0.75	0.57	0.40	0.53	0.61	0.36	0.53
Min Risk	0.97	0.71	0.46	0.69	0.76	0.39	0.62
Bounded Min Risk	0.97	0.71	0.47	0.67	0.78	0.42	0.63
Min Risk-Amb	0.95	0.70	0.44	0.67	0.76	0.41	0.62
Bounded Min Risk-Amb	0.94	0.70	0.45	0.66	0.76	0.41	0.63
Turnover							
No hedge	0%	0%	0%	0%	0%	0%	0%
Half hedge	32.11%	43.08%	39.85%	39.85%	47.16%	56.10%	54.28%
Full hedge	65.37%	86.78%	80.44%	80.08%	95.09%	115.30%	110.21%
Min Risk	130.76%	176.32%	171.16%	140.57%	203.35%	255.15%	229.54%
Bounded Min Risk	76.41%	109.07%	123.48%	101.28%	132.27%	168.67%	167.61%
Min Risk-Amb	73.71%	108.77%	129.68%	104.09%	133.09%	160.78%	159.15%
Bounded Min Risk-Amb	64.16%	95.44%	117.85%	92.14%	118.72%	154.87%	152.10%
Max Drawdown							
No hedge	26.13%	32.64%	48.21%	40.91%	28.97%	56.21%	48.39%
Half hedge	28.20%	34.44%	43.53%	37.48%	32.05%	52.39%	47.15%
Full hedge	32.14%	38.41%	38.89%	34.14%	39.07%	50.53%	46.76%
Min Risk	22.47%	30.90%	33.67%	26.78%	29.75%	49.13%	39.52%
Bounded Min Risk	20.32%	27.41%	28.88%	27.04%	26.33%	39.11%	34.18%
Min Risk-Amb	20.64%	27.34%	29.58%	25.32%	26.18%	40.70%	35.16%
Bounded Min Risk-Amb	21.36%	28.51%	30.03%	26.44%	27.29%	40.44%	35.68%

Table 1 – This table reports the out-of-sample back test volatility, Sharpe ratio, turnover and maximum drawdown of portfolios featuring different currency overlay models. We present results for equally weighted global mixed portfolios consisting of equity, bonds and cash. Columns represent base countries and rows represent the currency overlay strategies. First group of three models refers to the constant hedging strategies. Second group of two models refers to strategies in which the currency position is chosen optimally in order to minimize risk only (no model uncertainty). Third group of two models refers to a strategy of an agent with the robust (ambiguity adjusted) mean–variance preferences, the risk aversion parameter of $\lambda = 5$ and the ambiguity aversion parameter of $\theta = 8$. All presented results are computed with daily returns, hedging at a monthly horizon, and relative transaction costs of 5 basis points.

The largest increase in Sharpe ratio is observed for Australian investors, whose currency on average appreciated in comparison to other currencies and positively affected the realized average return. On the contrary, the changes in Sharpe ratios across different models for Japanese investors is limited. The reason for this is that Japanese yen on average depreciated in comparison to other currencies. It is also important to note that these results should be interpreted with caution because they are calculated using sample average currency returns, which are noisy estimates of the true mean currency returns.

Next part of Table 1 studies the turnover of different currency overlay strategies. We define the average (monthly) hedging turnover $\overline{\mathrm{HT}^s}$ of a strategy s as

$$\overline{\mathrm{HT}^s} = \frac{1}{T-1} \sum_{t=1}^{T-1} \sum_{c=2}^{k+1} |\phi_{t,c}^s|, \tag{29}$$

where $T-1$ is the number of trading days, k is the number of foreign currencies, and $\phi_{t,c}^s$ is the notional (relative to the total portfolio value) of a hedge of currency c at time t. Therefore, $\overline{\mathrm{HT}^s}$ indicates an average amount of trading for a hedging strategy s. In conjecture with relative transaction costs, the average cost of the strategy can then also be inferred.

The first notable difference in the turnover part of Table 1 is that the average turnovers of the fully hedged Australian and Japanese investors are 65% and 115%, respectively. This happens because the Australian investors on average receive positive pay-offs from their fully hedged currency forward positions, and consequently increase the domestic cash portfolio weight. The opposite happens to Japanese (and US) investors, who through hedging over time accumulate a negative position in domestic cash. This is the reason why average hedging turnover for a particular hedging strategy defined in Eq. (29) can vary substantially over different base currencies.

Observe the sizable turnover of the minimum risk hedging strategy. This is due to the fact of possible extreme positions in the estimated optimal currency exposure. Constraining the currency positions substantially lowers the turnover, which can be seen in the case of bounded models. Moreover, minimum risk and ambiguity strategy also massively lowers the realized average turnover compared to the minimum risk case. This comes from the fact that ridge regression induces a variance-bias trade-off, shrinks the estimated exposure towards zero, and in such way stabilizes the optimal exposure estimates over time.

The last part of Table 1 studies the realized maximum portfolio drawdown. A maximum drawdown is the maximal decline from a peak to a trough of portfolio cumulative returns, before a new peak is attained. It is an indicator of a downside risk over a time period of the back test. Observe that the constant hedging strategies produce contrasting outcomes. For some base currencies the downside risk is reduced via full hedging, and the opposite for other currencies. The downside risk is substantially increased for UK investor, from a maximum drawdown of 29% for no hedging to 39% in the case of full hedging. It is interesting to observe that the increase in Sharpe ratio for a fully hedged Australian investor comes at the expense of increased downside risk. Again, all currency overlay models outperform

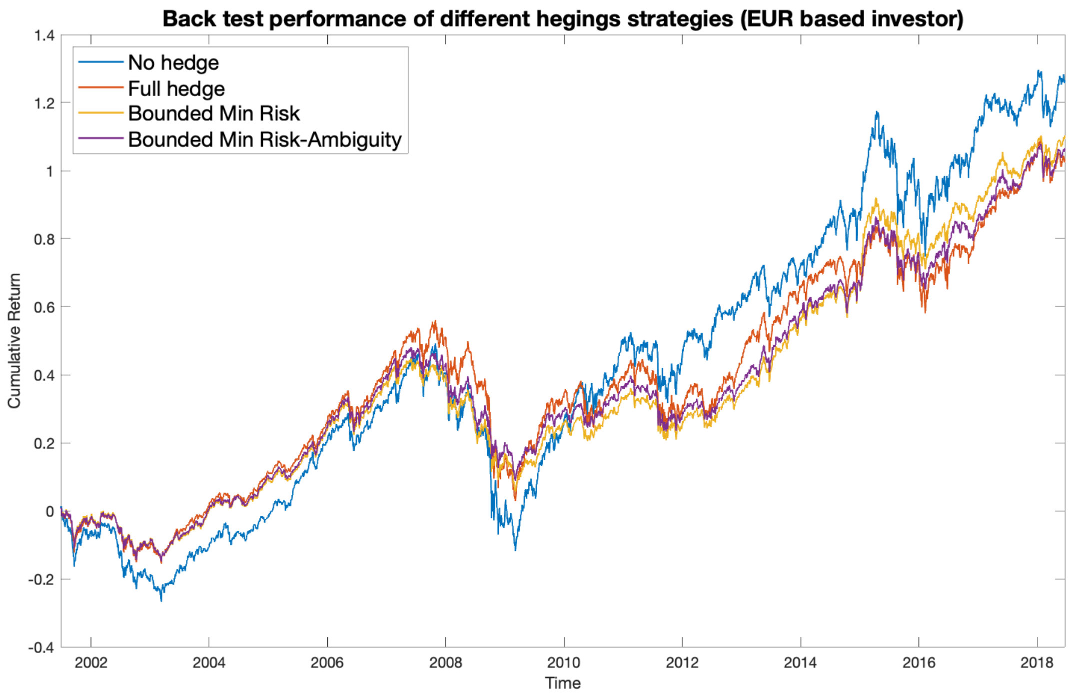

Figure 7 – Back test performance of different constant and model hedging strategies is plotted here. We consider a global equally weighted equity, bond and cash portfolio for a EUR based investor. We assume independent prediction models with $\mathrm{Var}_\mu[\,\mathrm{E}_\mathbb{Q}[\mathbf{e}_{t+1} - \mathbf{f}_t]\,] = \frac{1}{T^2}\mathbf{I}$ and the uncovered interest rate parity to hold.

constant hedging and provide considerable reduction of maximum portfolio drawdown across all base currencies. This result is especially important for pension funds where currency hedging decisions are predominantly driven by possible reductions in portfolio (downside) risk.

Figure 7 depicts the cumulative portfolio return for the EUR based investor. One can observe the high variability of the portfolio cumulative return in the case of no hedging. This variability is substantially decreased with full hedging of implicit portfolio currency exposure. The bounded minimum risk as well as bounded minimum risk and ambiguity models plotted for comparison show that the performance is stabilized and usually lies in between no and full hedging. However, as observed in Table 1, the performance of models outperforms constant hedging in terms of portfolio volatility, Sharpe ratio as well as portfolio drawdown. Note that the plotted cumulative returns are corrected for transaction costs. Considering the analysis of hedging turnover from above, one can observe that the costs arising from the currency overlay strategies are acceptable and make these strategies cost efficient. This occurs as hedging is performed solely with currency forward contracts which are relatively inexpensive and liquid instruments. Moreover, these results are also robust to an increase in transaction costs.

To conclude the out-of-sample back test analysis, let us inspect how the estimated value of optimal currency exposure (for a particular currency) varies over time. Figure 8 depicts the optimal exposure in JPY for a global equally weighted EUR based investor. The optimal exposure is expressed in relative

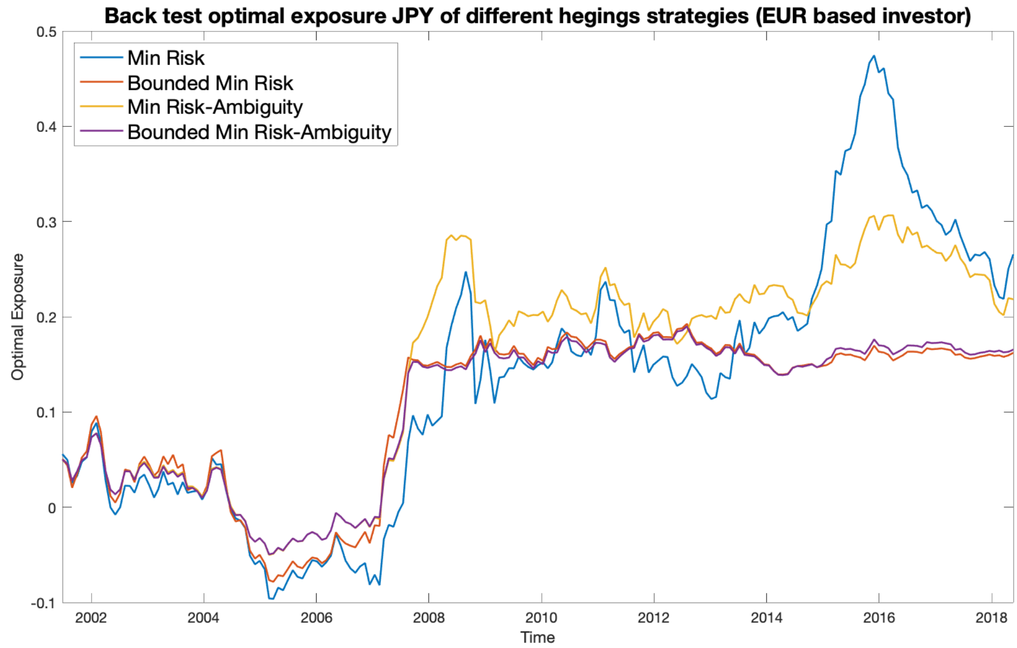

Figure 8 – Optimal currency exposure in JPY for different models over the time of back test is plotted here. We consider a global equally weighted equity, bond and cash portfolio for a EUR based investor. We assume independent prediction models with $\mathrm{Var}_\mu[\,\mathrm{E}_\mathbb{Q}[\mathbf{e}_{t+1} - \mathbf{f}_t]\,] = \frac{1}{T^2}\mathbf{I}$ and the uncovered interest rate parity to hold. Observe how the stability of optimal JPY exposure increases when one considers ambiguity in addition to risk. Moreover, the stability is additionally improved in the presence of exposure constrains, which can be seen in the bounded models.

terms with respect to the total portfolio value. All of the models roughly agree on the value of optimal exposure. This is where the similar results of the back test for the model based hedging comes from. However, one can observe that the currency exposure of the unbounded minimum risk investor can become very large. This occurs as leverage is in this case not constrained. This is alleviated via the minimum risk and ambiguity hedge, where the ridge regression shrinks the optimal exposure towards zero. We additionally stabilize the estimates via the constrains on possible exposures. In such way bounded model estimates become very stable over time, induce less hedging costs, and improve the empirical performance of the investigated currency overlay strategies.

5 Conclusion

The goal of this paper is to study the choice of optimal foreign currency exposure for an investor who is risk and ambiguity averse. We start with presenting a completely general framework for hedged portfolio returns in a model-free setting, meaning that there are no assumptions on the dynamics of asset or currency returns. In order to account for model uncertainty directly in an agent's currency allocation decision, we employ robust mean–variance preferences which explicitly capture the agent's risk and ambiguity aversion. We combine the expressions for hedged portfolio returns (which separate the returns arising from assets in their local currencies and the pure currency exposure) with the robust mean–variance utility representation and derive closed form expressions that characterize the optimal currency exposure for a risk-and-ambiguity-averse investor.

We examine the case of an investor whose portfolio consists of domestic assets—which she regards as purely risky—and is considering whether exposure to other currencies—which she regards as ambiguous—would help improve her risk-ambiguity-return spectrum. We show that in the limit, when the risk aversion parameter $\lambda \to \infty$, the optimal currency exposure converges to the minimum variance case (infinitely risk averse agent). On the other hand, when the ambiguity aversion parameter $\theta \to \infty$, the optimal foreign currency exposure converges to 0 (full hedging), and the investor holds an exposure only to the domestic (purely risky) position. This result shows that the puzzle of insufficient currency diversification (home currency bias) can be driven by investor's ambiguity aversion.

The paper shows that the optimal in-sample currency exposure for a risk-and-ambiguity-averse agent can be found by a generalized ridge regression of the demeaned hedged portfolio returns on the demeaned currency excess returns and shrunk towards the infinitely ambiguity averse optimal exposure distorted by the level of model uncertainty. Such artificial regression recovers the sample efficient currency exposure, highlights the stochastic nature of optimal exposure calculated from a finite sample, empowers a geometric interpretation of obtained results, and enables inference procedures for hypotheses about the efficient currency exposure. The main reasons for the geometric interpretation of the optimal currency exposure as a solution to the described regression problem are the linear relationship between the currency exposure and the portfolio return (derived in a model-free setting) and the robust mean–variance preferences.

The generalized penalty term of the ridge regression corresponds to the utility loss arising from the model uncertainty. This result emphasizes the importance of predictive modelling and risk management for international investors. Moreover, in the first best, the case without model uncertainty, the projection of hedged portfolio returns onto the space spanned by the currency excess returns is orthogonal. With the penalization, which arises from model uncertainty, shrinkage is induced. The magnitude of ambiguity aversion parameter (penalizing factor) and the structure of model uncertainty control the distance and direction of shrinkage and in such way characterize the solution of the second best (in the presence of model uncertainty). This regularization, biasing of the estimate which stabilizes the inference, is here not assumed a priori. It originates as a solution to the robust (ambiguity consistent) mean–variance maximization problem. In such way we formally relate the areas of financial economics and statistical learning.

In the empirical part of the work we start with investigating the impact of ambiguity aversion on optimal currency allocations. Using the non-parametric bootstrap, we show that in the case of no uncertainty, $\theta = 0$, the distribution of the optimal currency exposure estimator is extremely wide (exhibits large parameter uncertainty). For larger values of θ its distribution shifts towards the ridge target (which is equal to zero in our example), and narrows extensively. This corresponds to an increase in bias and a simultaneous shrinkage of confidence intervals. Building upon the intuition of this bias-variance trade-off we show that acknowledging uncertainty can, for particular values of λ and θ, lead to an improved estimator of optimal currency exposure, measured in the sense of mean squared error.

An out-of-sample back test is implemented, where the performance of the derived theoretical model on the historical market data ranging from 1999 to 2018 is analyzed. We find that the model based hedging substantially outperforms the constant hedging strategies net of transaction costs. The improvements can be seen as reductions in portfolio volatility, improved Sharpe ratios as well as reductions in maximum drawdowns. Ambiguity induces shrinkage and in such way stabilizes optimal currency exposure estimations over the back test. This yields a reduction in average hedging turnover and shows that acknowledging ambiguity outperforms other currency hedging strategies.

This working paper allows for possible generalizations in several research directions. Let us start with the theoretical considerations. Variance is a natural benchmark, which we extended by accounting for model uncertainty. Further implications of risk and ambiguity in higher moments can be pursued. Such higher moments can then be accounted for by different risk measures which can also explicitly capture the downside risk which is, for example, often present in emerging market currencies. This would require a derivation of robust preferences capturing also the co-skewness and co-kurtosis of asset and currency returns, from where one could study their effect on the optimal currency exposure. Moreover, as forward contracts (used for hedging in the current setup) are linear instruments, one can as well study hedging with options, especially to investigate the effect of mitigating the currency downside risk. The setup would in this case become non-linear, which requires more complicated optimization framework and its solutions could potentially not be presented in closed form. It is also possible to integrate the speculative and risk management components of currency demand by solving the optimal portfolio choice problem for an investor choosing currency positions jointly with stock and bond positions (not treating portfolio weights as given such as in this work). This would lead to a portfolio choice which is efficient, in comparison to determining the asset weights separately from underlying currency exposure in a two-step optimization procedure. One could also include the effect of hedging (trading) transaction costs directly to the optimized utility function. This would correspond to an investor seeking to improve the risk-ambiguity-return characteristics in a cost-efficient manner. The transaction costs would in this case correspond to the L^1-normed penalty (Lasso regression), such as in Brodie, Daubechies, De Mol, Giannone, and Loris (2009). The generalized ridge regression would in such case be extended to the generalized elastic net regularization.

From the empirical point of view, it would be interesting to look at emerging market currencies and indices jointly with the already investigated developed markets. It is also possible to extend an empirical analysis using different asset classes, such as corporate bonds, and investigate a possible difference in optimal currency exposure of the portfolio consisting of investment grade vs high yield bonds of developed and emerging markets. A potential prospect is also an inclusion of commodities (for example broad commodity indices) or studying specific sectors, such as energy, agriculture, precious metals, and observing if different sectors in various market environments covary with currencies in different ways. This relates to the growing academic interest in sector rotation investment strategies, where one can examine how currencies relate to the corresponding macro variables used for predictions and sector rotation decisions. One can as well look at different investment styles, such as value, size, momentum, minimum variance etc., and investigate the relation between currency exposure and different types of factors (factor models). Such analysis can be performed out-of-sample with different sampling frequencies and maturities of hedges in order to investigate the robustness of the proposed currency overlay strategies. The effects of specific events, such as Swiss franc unpeg and Brexit, can be investigated as well.

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Appendix

Derivation of Eq. (22)

The main idea of the following derivation is to start with Eq. (18) and use matrix algebra in order to rewrite the expression into (22).

The optimal exposure for a risk-and-ambiguity-averse international investor is given in (18) as

$$\begin{aligned}\boldsymbol{\Psi}_t^* = &-\Big(\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,] + \theta \mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,]\,]\Big)^{-1}\cdot \\ &\cdot\Big(\lambda \mathrm{Cov}_{\bar{\mathbb{Q}}}[\,\tilde{R}^{fh}_{\mathcal{P},t+1}, \mathbf{e}_{t+1}-\mathbf{f}_t\,] + \theta \mathrm{Cov}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\tilde{R}^{fh}_{\mathcal{P},t+1}\,], \mathrm{E}_{\mathbb{Q}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,]\,] - \mathrm{E}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,]\Big),\end{aligned}$$

Assuming positive definiteness of $\mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t]\,]$, one can rearrange the terms as

$$\begin{aligned}\boldsymbol{\Psi}_t^* = \quad &-\Big(\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,] + \theta \mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,]\,]\Big)^{-1} \\ &\cdot\Big(\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,](\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,])^{-1}(\lambda \mathrm{Cov}_{\bar{\mathbb{Q}}}[\,\tilde{R}^{fh}_{\mathcal{P},t+1}, \mathbf{e}_{t+1}-\mathbf{f}_t\,] - \mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1}-\mathbf{f}_t]) \\ &+\theta \mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,]\,](\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,])^{-1}(\lambda \mathrm{Cov}_{\bar{\mathbb{Q}}}[\,\tilde{R}^{fh}_{\mathcal{P},t+1}, \mathbf{e}_{t+1}-\mathbf{f}_t\,] - \mathrm{E}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,]) \\ &-\theta \mathrm{Var}_{\mu}(\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_t])(\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,])^{-1}(\lambda \mathrm{Cov}_{\bar{\mathbb{Q}}}[\,\tilde{R}^{fh}_{\mathcal{P},t+1}, \mathbf{e}_{t+1}-\mathbf{f}_t\,] - \mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1}-\mathbf{f}_t]) \\ &+\theta \mathrm{Cov}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\tilde{R}^{fh}_{\mathcal{P},t+1}\,], \mathrm{E}_{\mathbb{Q}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,]\,]\Big) \\ = \quad &-\Big(\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,] + \theta \mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,]\,]\Big)^{-1} \\ &\cdot\Big((\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,] + \theta \mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,]\,]) \\ &\cdot(\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,])^{-1}(\lambda \mathrm{Cov}_{\bar{\mathbb{Q}}}[\,\tilde{R}^{fh}_{\mathcal{P},t+1}, \mathbf{e}_{t+1}-\mathbf{f}_t\,] - \mathrm{E}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,]) \\ &-\theta \mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,]\,](\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,])^{-1}(\lambda \mathrm{Cov}_{\bar{\mathbb{Q}}}[\,\tilde{R}^{fh}_{\mathcal{P},t+1}, \mathbf{e}_{t+1}-\mathbf{f}_t\,] - \mathrm{E}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,]) \\ &+\theta \mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,]\,]\mathrm{Var}_{\mu}(\mathrm{E}_{\mathbb{Q}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,])^{-1}\mathrm{Cov}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\tilde{R}^{fh}_{\mathcal{P},t+1}\,], \mathrm{E}_{\mathbb{Q}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,]\,]\Big) \\ = \quad &-\mathrm{Var}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,]^{-1}(\mathrm{Cov}_{\bar{\mathbb{Q}}}[\,\tilde{R}^{fh}_{\mathcal{P},t+1}, \mathbf{e}_{t+1}-\mathbf{f}_t\,] - \frac{1}{\lambda}\mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1}-\mathbf{f}_t]) \\ &+\big(\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,] + \theta \mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,]\,]\big)^{-1}\theta \mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,]\,] \\ &\cdot\big(\mathrm{Var}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,]^{-1}(\mathrm{Cov}_{\bar{\mathbb{Q}}}[\,\tilde{R}^{fh}_{\mathcal{P},t+1}, \mathbf{e}_{t+1}-\mathbf{f}_t\,] - \frac{1}{\lambda}\mathrm{E}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,]) \\ &-\mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,]\,]^{-1}\mathrm{Cov}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\tilde{R}^{fh}_{\mathcal{P},t+1}\,], \mathrm{E}_{\mathbb{Q}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,]\,]\big) = \\ = \quad &\boldsymbol{\Psi}^*_{t,mv} + \big(\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,] + \theta \mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,]\,]\big)^{-1}\theta \mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\mathbf{e}_{t+1}-\mathbf{f}_t\,]\,]\,(\boldsymbol{\Psi}^*_{t,amb} - \boldsymbol{\Psi}^*_{t,mv}),\end{aligned}$$

where we used

$$\begin{aligned}
\boldsymbol{\Psi}^{*}_{t,amb} &= -\mathrm{Var}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1}-\mathbf{f}_{t}]\,]^{-1}\mathrm{Cov}_{\mu}[\,\mathrm{E}_{\mathbb{Q}}[\,\tilde{R}^{fh}_{\mathcal{P},t+1}\,],\mathrm{E}_{\mathbb{Q}}[\,\mathbf{e}_{t+1}-\mathbf{f}_{t}\,]\,],\\
\boldsymbol{\Psi}^{*}_{t,mv} &= -\mathrm{Var}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_{t}\,]^{-1}\Big(\mathrm{Cov}_{\bar{\mathbb{Q}}}[\,\tilde{R}^{fh}_{\mathcal{P},t+1},\mathbf{e}_{t+1}-\mathbf{f}_{t}\,]-\frac{1}{\lambda}\mathrm{E}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_{t}\,]\Big)=\\
&= \boldsymbol{\Psi}^{*}_{t,risk}+\frac{1}{\lambda}\mathrm{Var}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_{t}\,]^{-1}\mathrm{E}_{\bar{\mathbb{Q}}}[\,\mathbf{e}_{t+1}-\mathbf{f}_{t}\,].
\end{aligned}$$

This concludes the derivation of Eq. (22).

Derivation of Eq. (27)

The main idea of the derivation is to show that the solution to the generalized ridge regression problem is of the form of first equality in (27), and then to use matrix algebra and rewrite the expression into the form of second equality of (27).

The aim of this derivation is to obtain a closed form solution to the in-sample risk and ambiguity adjusted currency exposure optimization problem. In order to achieve this, we have to solve the generalized ridge regression given in Eq. (26). We use the same notation as in the main part of the paper. Hence, let $\mathbf{X}$ denote the $(T \times n)$ matrix of demeaned historical currency excess returns $\mathbf{e}_{t+1} - \mathbf{f}_t$, and let $\mathbf{y}$ denote the $(T \times 1)$ vector of demeaned historical fully hedged portfolio return R_t^{fh}, where T is the number of observations in the sample. Moreover, let $\mathbf{W} = \frac{\lambda}{T}\mathbf{I}$, where $\mathbf{I}$ is a $(T \times T)$ identity matrix, $\mathbf{Z} = \theta \mathrm{Var}_\mu(\mathrm{E}_\mathbb{Q}[R_{t+1}^h])$ and $\mathbf{z_0} = -\mathbf{\Psi}_{t,amb}^*$, where we assumed that the inverse of $\mathrm{Var}_\mu(\mathrm{E}_\mathbb{Q}[\mathbf{e}_{t+1} - \mathbf{f}_t])$ exists. By Eq. (26) we deal with an optimization problem of the form

$$\begin{aligned}\arg\min_{\mathbf{\Psi}_t} \mathcal{L}_\mathbb{H}(R_{t+1}^h) &= \arg\min_{\mathbf{\Psi}_t} \left\{ \Big(\mathbf{y} + \mathbf{X}\mathbf{\Psi}_t\Big)' \mathbf{W} \Big(\mathbf{y} + \mathbf{X}\mathbf{\Psi}_t\Big) + (\mathbf{\Psi}_t + \mathbf{z_0})' \mathbf{Z} (\mathbf{\Psi}_t + \mathbf{z_0}) + \text{rest} \right\} = \\ &= \arg\min_{\mathbf{\Psi}_t} \big\|\mathbf{y} - \mathbf{X}(-\mathbf{\Psi}_t)\big\|_\mathbf{W}^2 + \big\|(-\mathbf{\Psi}_t) - (-\mathbf{\Psi}_{t,amb}^*)\big\|_\mathbf{Z}^2,\end{aligned}$$

where we explicitly write the terms which depend on $\mathbf{\Psi}_t$ and with rest denote other terms which do not affect the optimization. Using Eq. (25) this can be written as

$$\arg\min_{\mathbf{\Psi}_t} \mathcal{L}_\mathbb{H}(R_{t+1}^h) = \arg\min_{\mathbf{\Psi}_t} \left\{ \mathbf{\Psi}_t{}' \mathbf{X}' \mathbf{W} \mathbf{X} \mathbf{\Psi}_t + 2\mathbf{\Psi}_t{}' \mathbf{X}' \mathbf{W} \mathbf{y} + \mathbf{\Psi}_t{}' \mathbf{Z} \mathbf{\Psi}_t + 2\mathbf{\Psi}_t{}' \mathbf{Z} \mathbf{z_0} + \text{rest} \right\}.$$

A vector derivative with respect to $\mathbf{\Psi}_t{}'$ then yields the first-order optimality condition given by

$$2\mathbf{X}' \mathbf{W} \mathbf{X} \mathbf{\Psi}_t + 2\mathbf{X}' \mathbf{W} \mathbf{y} + 2\mathbf{Z}\mathbf{\Psi}_t + 2\mathbf{Z}\mathbf{z_0} \stackrel{!}{=} 0.$$

Rearranging the terms shows the first equality in Eq. (27), given by

$$\widehat{\mathbf{\Psi}}_{t,\mathbb{H}}^* = -\big(\mathbf{X}' \mathbf{W} \mathbf{X} + \mathbf{Z}\big)^{-1} \big(\mathbf{X}' \mathbf{W} \mathbf{y} + \mathbf{Z}\mathbf{z_0}\big).$$

Note that the second-order vector derivative is positive definite, which ensures that the unique global minimum of the loss function is attained. Now we can rewrite the obtained equation as

$$\begin{aligned}\widehat{\mathbf{\Psi}}_{t,\mathbb{H}}^* &= -\big(\mathbf{X}' \mathbf{W} \mathbf{X} + \mathbf{Z}\big)^{-1} \big(\mathbf{X}' \mathbf{W} \mathbf{X} (\mathbf{X}' \mathbf{W} \mathbf{X})^{-1} \mathbf{X}' \mathbf{W} \mathbf{y} + \\ &\quad + \mathbf{Z}(\mathbf{X}' \mathbf{W} \mathbf{X})^{-1} \mathbf{X}' \mathbf{W} \mathbf{y} - \mathbf{Z}(\mathbf{X}' \mathbf{W} \mathbf{X})^{-1} \mathbf{X}' \mathbf{W} \mathbf{y} + \mathbf{Z}\mathbf{z_0}\big) = \\ &= -(\mathbf{X}' \mathbf{W} \mathbf{X})^{-1} \mathbf{X}' \mathbf{W} \mathbf{y} + (\mathbf{X}' \mathbf{W} \mathbf{X} + \mathbf{Z})^{-1} \mathbf{Z}((\mathbf{X}' \mathbf{W} \mathbf{X})^{-1} \mathbf{X}' \mathbf{W} \mathbf{y} - \mathbf{z_0}) = \\ &= \widehat{\mathbf{\Psi}}_{t,\mathbb{H},risk}^* + \big(\mathbf{X}' \mathbf{W} \mathbf{X} + \mathbf{Z}\big)^{-1} \mathbf{Z} \left(\mathbf{\Psi}_{t,amb}^* - \widehat{\mathbf{\Psi}}_{t,\mathbb{H},risk}^*\right),\end{aligned}$$

where we used $\mathbf{z_0} = -\mathbf{\Psi}^*_{t,amb}$ and the definition of $\mathbf{W}$ in order to express

$$\widehat{\mathbf{\Psi}}^*_{t,\mathbb{H},risk} = -(\mathbf{X}'\,\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\,\mathbf{W}\mathbf{y} = -(\mathbf{X}'\,\mathbf{X})^{-1}\mathbf{X}'\,\mathbf{y}.$$

This concludes the derivation of Eq. (27).

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c/o University of Geneva, Bd. Du Pont d'Arve 42, CH-1211 Geneva 4
T +41 22 379 84 71, rps@sfi.ch, www.sfi.ch